EXCEEDINGLY DERANGING !

ROBERTO MANTACI AND FANJA RAKOTONDRAJAO

Abstract. This paper studies the exceedance distribution on derangements and gives its generating function. It also studies the exceedance distribution on the set of even derangements and the set of odd derangements and proves combinatorially an unexpected relation between the number of odd and the number of even derangements having a given number of exceedances.

Résumé. Nous étudions la distribution des excédances sur les dérangements et nous donnons la fonction génératrice de cette distribution. Nous étudions également la distribution de cette statistique sur les dérangements paires et impaires et demontrons une relation inattendue entre le nombre de dérangements paires et impaires ayant un nombre fixé d'excédances.

1. Introduction and preliminairies

While the study of the so-called Eulerian statistics (*descents, exceedances, readings*) over the symmetric group is a well-known subject (see [Fo-Schu], [Ga-Ge], [Ge-Re], [Kn], [Ri]), the study of these same statistics on the set of derangements is a newer one.

Some results about the distribution of these and other statistics like inversions and major index over the set of derangements are available, but they are seldom proved using the derangements themselves. Indeed, they often make use of correspondences between derangements and other combinatorial objects, on which the identities are studied and proven.

For instance, Garsia and Remmel ([Ga-Re]), gave a formula for the q-derangement numbers (that is, polynomials in the variable q that q-count the derangements by major index) as a specialization of a generating function counting permutations by descents, major index, and cycle structure, and they proved this result using a correspondence between partitions and permutations.

q-Derangement numbers have also been studied by other authors ([De], [De-Wa], [Wa]), who have given a combinatorial interpretation, sometimes based on bijections that associate derangements with another set of permutations.

We will give in this paper the distribution of the exceedance statistic over derangements by studying it directly on derangements. The numbers $a_{n,k}$ that count the derangements over n objects having k exceedances, prove to be symmetric like the Eulerian numbers. We will give an exponential generating function for these numbers as well as a recursive relation defining them.

We will also study the distribution of exceedances on even and on odd derangements and will prove that the number of odd derangements having k exceedances and the number of even derangements having k exceedances differ by 1 for all n and k. We give a combinatorial proof of this result.

The proof provided for this result is a proof "by case". It can certainly be argued that such proofs are not "elegant" and we certainly hope that a nicer proof of this result can be found in the future. However, if we dare use the words of the second author : "derangements are like a rose bush, when you touch them, you need to be careful of thorns." It could be added that derangements are fragile and that, when you manipulate them, each of them needs a special care and treatment in order to preserve the property of being a derangement. For the proof in discussion, we needed to introduce

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a (quasi) bijective map that preserves the property of being a derangement, preserves the number of exceedance (it turns out that, indeed, our map preserves the *set* of exceedances) and exchanges parity. It is then not very surprising that a definition of such map "by case" is needed.

Let us denote by $[n]$ the interval $\{1, 2, 3, ..., n\}$, by σ a permutation of the symmetric group $\mathfrak{S}_n$, by A_n the alternating group of rank n, that is, the group of all even permutations.

Definition 1.1. *We will say that σ presents an exceedance (resp. an anti-exceedance) in $i \in [n]$ if $\sigma(i) > i$ (resp. $\sigma(i) \leq i$). In this case we will say that i is an exceedant (resp. an anti-exceedant).*

Definition 1.2. *We will say that $i \in [n]$ is a fixed point for σ if $\sigma(i) = i$.*

Definition 1.3. *We will say that σ is a derangement if σ has no fixed points.*

We will denote by D_n the set of all derangements of $\mathfrak{S}_n$ and by d_n the integer $|D_n|$.
We will denote by $a_{n,k}$ the cardinality of $D_{n,k}$ the set of the derangements having k exceedances, by $exc(\sigma)$ the number of the exceedances of σ, and by $cyc(i)$ the length of the cycle containing i in the disjoint cycle decomposition of σ.

2. Bijections between D_n and $[n-1] \times (D_{n-1} \bigcup D_{n-2})$

It is well known that the number d_n of derangements satisfies the following identity :

$$d_n = (n-1)(d_{n-1} + d_{n-2}).$$

This result can be found in the article of Remmel, [Re], who also gives the suggestion of a bijection that maps the set $[n-1] \times D_{n-1}$ onto the derangements of $[n]$ such that n lies in a cycle of length greater than or equal to 3 and maps the set $[n-1] \times D_{n-2}$ onto the derangements of $[n]$ such that n lies in a cycle of length 2.

It is possible to show several bijections between these two sets that would prove this identity, the authors of this paper for instance have noticed already two of them. We would like to give the explicit definition of one of them, because we intend to use the idea behind this bijection to show our next result.

Let $\psi : ([n-1] \times (D_{n-1} \bigcup D_{n-2})) \to D_n$ be defined as follows:

- if $(i, \delta') \in [n-1] \times D_{n-2}$, then $\psi(i, \delta')$ is the permutation obtained from the cycle decomposition of δ' by increasing all the integers greater than or equal to i by 1 and by adding the new cycle $(i\ n)$.
- if $(i, \delta') \in [n-1] \times D_{n-1}$, then $\delta = \psi(i, \delta')$ is the permutation such that:

$$\delta(j) = \begin{cases} n & if \quad j = i \\ \delta'(i) & if \quad j = n \\ \delta'(j) & if \quad j \neq i, n. \end{cases}$$

  In other terms, δ is obtained from δ' by inserting the integer n in the cycle containing i after i itself, e.g if $i = 3$ and $\delta' = (135)(24)$ then $\delta = (1365)(24)$.

Proposition 2.1. *The map ψ is a bijection from $([n-1] \times (D_{n-1} \bigcup D_{n-2}))$ to D_n.*

Proof. Notice that the image $\psi([n-1] \times D_{n-1})$ is contained in the set of all derangements of D_n in which n lies in a cycle of length greater than or equal to 3, while the image $\psi([n-1] \times D_{n-2})$ is contained in the set of all derangements of D_n in which n lies in a cycle of length 2. So we only need to show that there exists a map ϕ that

- associates an element of $([n-1] \times D_{n-2})$ with every derangement of D_n in which n lies in a cycle of length 2,
- associates an element of $([n-1] \times D_{n-1})$ with every derangement of D_n in which n lies in a cycle of length greater than or equal to 3 and

- is the inverse of ψ.

It is straightforward to verify that ϕ is the map defined as follows. If δ is a derangement in which n lies in a cycle of length greater than or equal to 3, then $\phi(\delta)$ is the couple (i,δ') where $i=\delta^{-1}(n)$ and for all integer j with $1\leq j\leq n-1$:

$$\delta'(j)=\begin{cases}\delta(j) & \textit{if} \quad j\neq i,n\\ \delta(n) & \textit{if} \quad j=i\end{cases}$$

In this case, the permutation δ' is clearly a derangement of D_{n-1}, since it is obtained by removing n from a derangement of D_n where n lies in a cycle of length greater than or equal to 3.
If δ is a derangement in which n lies in a cycle of length 2, then $\phi(\delta)$ is the couple (i,δ') where $i=\delta^{-1}(n)$ and δ' is obtained from the cylce decomposition of δ by removing the 2 cycle containing n and by decreasing by 1 all integers between $i+1$ and $n-1$.

In this case, the permutation δ' is clearly a derangement of D_{n-2}. □

3. The exceedance distribution on derangements

We will give a recursive relation for the exceedance distribution on derangements and give its generating function. The first values of the numbers

$$a_{n,k}=card\{\delta\in D_n \mid \delta \text{ has } k \text{ exceedances }\}$$

are given in the following table:

	$a_{n,k}$				
	$k=1$	2	3	4	5
$n=1$	0				
2	1				
3	1	1			
4	1	7	1		
5	1	21	21	1	
6	1	51	161	51	1

Proposition 3.1. *The numbers $a_{n,k}$ are symmetric i.e.*

$$a_{n,k}=a_{n,n-k}$$

for all positive integers n, and $1\leq k\leq n-1$.

Proof. Let δ be a derangement in D_n having exactly k exceedances. The bijective map $\delta\mapsto\delta^{-1}$, associates δ with a derangement having exactly $n-k$ exceedances. □

Theorem 3.2. *The numbers $a_{n,k}$ satisfy the recurrence relation:*

$$a_{n,k}=(n-1)a_{n-2,k-1}+(n-k)a_{n-1,k-1}+ka_{n-1,k}, \tag{1}$$

for all integers $n\geq 2$ and $1\leq k\leq n-1$ with the initial conditions $a_{0,0}=1$ and $a_{1,0}=0$.

Proof. We will use the same idea of the bijection ψ described in the previous section.

- If δ' is a derangement of D_{n-2} having $k-1$ exceedances and i is any integer with $1\leq i\leq n-1$, then the permutation δ obtained from the cycle decomposition of δ' by increasing the integers that are greater or equal to i by 1 and by adding the new cycle $(i\ n)$, is a derangement of D_n having k exceedances. Indeed, any exceedance of δ' corresponds to an exceedance of δ and, moreover, δ also presents a new exceedance at i, since $\delta(i)=n$.
- If δ' is a derangement of D_{n-1} having $k-1$ exceedances and i is one of the $(n-1)-(k-1)=n-k$ anti-exceedants, then the permutation δ obtained from the cycle decomposition of δ' by inserting the integer n in the cycle containing i after i itself is a derangement of D_n having k exceedances. Indeed, if δ' has an exceedance in the integer j, then so does δ, moreover δ presents a further exceedance in i since $\delta(i)=n$.

- If δ' is a derangement of D_{n-1} having k exceedances and i is one of the k exceedants, then the permutation δ obtained from the cycle decomposition of δ' by inserting the integer n in the cycle containing i after i itself is again a derangement of D_n having k exceedances. Indeed, δ' and δ have exactly the same set of exceedances.

It is easy to check that the derangements of $D_{n,k}$ obtained this way are all distinct and that any derangement of $D_{n,k}$ can be obtained this way. Therefore this is a bijection between the set $D_{n,k}$ and the set

$$\begin{array}{ccl}([n-1]\times D_{n-2}) & \bigcup & \{(i,\delta') \mid \delta' \in D_{n-1,k-1},\ i \text{ is an anti-exceedant for } \delta'\} \\ & \bigcup & \{(i,\delta') \mid \delta' \in D_{n-1,k},\ i \text{ is an exceedant for } \delta'\}.\end{array}$$

Then, the cardinalities of $D_{n,k}$ and this set are equal, which concludes the proof. ☐

Proposition 3.3. *The exceedance distribution and the anti-exceedance distribution on derangements are the same.*

Proof. It follows from the symmetry outlined in Proposition 3.1. ☐

3.1. **The generating function of the exceedance distribution on derangements.** Let us denote by $a(x,u)$ the exponential generating function of the exceedance distribution on derangements, that is:

$$a(x,u) = \sum_{n\geq 0}\sum_{\delta\in D_n} x^{exc(\delta)}\frac{u^n}{n!}.$$

Proposition 3.4. *The generating function $a(x,u)$ of the exceedances on derangements satisfies the following differential equation:*

$$xua(x,u) = (1-xu)\frac{\partial}{\partial u}a(x,u) + x(x-1)\frac{\partial}{\partial x}a(x,u)$$

with the initial conditions

$$a(1,u) = D(u) = \frac{e^{-u}}{1-u} \ \ \textit{and} \ \ a(1,0) = 1.$$

Proof. The differential equation can be easily derived from the recurrence relation (1) given in Theorem 3.2. The initial condition:

$$a(1,u) = D(u) = \frac{e^{-u}}{1-u}$$

is due to the fact that when we set $x = 1$ the resulting formal series is the well-known generating function for the derangements. ☐

Theorem 3.5. *The generating function $a(x,u) = \sum_{n\geq 0}\sum_{\delta\in D_n} x^{exc(\delta)}\frac{u^n}{n!}$ of the exceedance distribution on derangements has the following closed form:*

$$a(x,u) \ = \ \frac{(1-x)e^{-xu}}{1-xe^{(1-x)u}}.$$

Proof. The function $\frac{(1-x)e^{-xu}}{1-xe^{(1-x)u}}$ satisfies the differential equation given in the previous proposition, as well as the initial conditions. ☐

4. Derangements, exceedances and parity

Let us denote by i_n the cardinality of the set of the odd derangements and by p_n the cardinality of the set of the even derangements.

Proposition 4.1. *For all integer n one has:*

$$i_n = (n-1)(p_{n-1} + p_{n-2})$$

and

$$p_n = (n-1)(i_{n-1} + i_{n-2}),$$

respectively, with initial values $i_0 = 0$, $i_1 = 0$ and $p_0 = 1$, $p_1 = 0$.

Proof. The bijection ψ described in Prop.2.1 associates to a couple (i, δ') with δ' odd derangement of D_{n-1} an even derangement $\delta \in D_n$ and vice-versa. $\square$

The first values of the numbers i_n and p_n are given in the following table:

n	1	2	3	4	5	6
i_n	0	1	0	6	20	135
p_n	0	0	2	3	24	130

4.1. **The exceedance distribution by parity.** Let us denote by $i_{n,k}$ the cardinality of the set of the odd derangements having k exceedances and by $p_{n,k}$ the cardinality of the set of the even derangements having k exceedances. The following tables report the values of these numbers for $1 \leq n \leq 6$ and $1 \leq k \leq 5$.

$i_{n,k}$					
	$k=1$	2	3	4	5
$n=1$	0				
2	1				
3	0	0			
4	1	4	1		
5	0	10	10	0	
6	1	26	81	26	1

$p_{n,k}$					
	$k=1$	2	3	4	5
$n=1$	0				
2	0				
3	1	1			
4	0	3	0		
5	1	11	11	1	
6	0	25	80	25	0

We have the following results:

Proposition 4.2. *For all positive integers n and k, with $1 \leq k \leq n-1$, one has:*

$$i_{n,k} = (n-1)p_{n-2,k-1} + (n-k)p_{n-1,k-1} + kp_{n-1,k}$$

and

$$p_{n,k} = (n-1)i_{n-2,k-1} + (n-k)i_{n-1,k-1} + ki_{n-1,k}$$

with the respective initial conditions $i_{0,0} = 0$, $i_{1,0} = 0$ and $p_{0,0} = 1$, $p_{1,0} = 0$.

Proof. The process described in Theorem 3.2 to prove the recursive formula for the numbers $a_{n,k}$ allows to construct an odd derangement of D_n starting from an even one of D_{n-1} or of D_{n-2}, and vice-versa. $\square$

Let us draw your attention on the first values of the differences $i_{n,k} - p_{n,k}$.

$i_{n,k} - p_{n,k}$					
	$k=1$	2	3	4	5
$n=1$	0				
2	1				
3	-1	-1			
4	1	1	1		
5	-1	-1	-1	-1	
6	1	1	1	1	1

Proposition 4.3. *For all positive integers n and for all k with $1 \leq k \leq n-1$, one has:*

$$i_{n,k} - p_{n,k} = (-1)^n.$$

Proof. A recursive proof of this proposition follows from the recurrence relations satisfied by the numbers $i_{n,k}$ and $p_{n,k}$ given in Proposition 4.1, while a combinatorial proof of this result is the subject of the next separate section of this paper. □

Corollary 4.4. *For all positive integer n, one has:*

$$i_n - p_n = (-1)^n(n-1).$$

4.2. **The generating function of the $i_{n,k}$ and $p_{n,k}$.**

Proposition 4.5. *The generating function*

$$i(x,u) = \sum_{n\geq 0} \sum_{\substack{\delta\in D_n \\ \delta\notin A_n}} x^{exc(\delta)} \frac{u^n}{n!}$$

of the exceedances distribution on the odd derangements has the closed form :

$$i(x,u) \quad = \quad \frac{1}{2}\left(\frac{(1-x)e^{-xu}}{1-xe^{(1-x)u}} + \frac{xe^{-u}-e^{-xu}}{(1-x)}\right).$$

The generating function

$$p(x,u) = \sum_{n\geq 0} \sum_{\delta\in D_n\cap A_n} x^{exc(\delta)} \frac{u^n}{n!}$$

of the exceedances distribution on the even derangements has the closed form :

$$p(x,u) \quad = \quad \frac{1}{2}\left(\frac{(1-x)e^{-xu}}{1-xe^{(1-x)u}} - \frac{xe^{-u}-e^{-xu}}{(1-x)}\right).$$

Proof. The series $\displaystyle\sum_{n\geq 0}(-1)^n \sum_{i=1}^{n-1} x^i \frac{u^n}{n!}$ has the closed form: $\displaystyle\frac{xe^{-u}-e^{-xu}}{1-x}$. Therefore, the two functions $i(x,u)$ and $p(x,u)$ are solutions of the system

$$\begin{cases} i(x,u)+p(x,u) = a(x,u) & = & \dfrac{(1-x)e^{-xu}}{1-xe^{(1-x)u}} \\ i(x,u)-p(x,u) & = & \dfrac{xe^{-u}-e^{-xu}}{1-x} \end{cases}$$

□

5. A combinatorial proof of the relation $i_{n,k} - p_{n,k} = (-1)^n$

Let ϕ be the map on the set D_n onto itself, defined in the following way:

(1) If $\delta(n) \neq 1$ and $\delta(1) \neq n$, then the derangement $\delta' = \phi(\delta)$ is obtained by multiplying the derangement δ on the right by the transposition $(\delta(1)\ \delta(n))$. In other terms, δ' is obtained from δ by exchanging the images of the integers 1 and n.
For example, if we take the derangement $\delta = 6\ 5\ 1\ 2\ 4\ 7\ 3$, then the derangement δ' image of δ by ϕ is equal to 3 5 1 2 4 7 6 and vice-versa.

(2) If $\delta(n) = 1$, then let i be the largest anti-exceedant of δ, with $i \neq n$ (if such an integer exists). The derangement $\delta' = \phi(\delta)$ is obtained by multiplying the derangement δ on the right by the transposition $(i\ i+1)$. In other terms, δ' is obtained from δ by exchanging the integers i and $i+1$ in the word $\delta(1)\cdots\delta(n)$.
For exemple, if the derangement $\delta = 2\ 8\ 9\ 3\ 7\ 5\ 4\ 6\ 1$, then the largest anti-exceedant is the integer 8 and the derangement δ' is equal to 2 9 8 3 7 5 4 6 1.

(3) If $\delta(1) = n$ and $\delta(n) \neq 1$, we have to distinguish the two following cases:
   (a) If $\delta(n) \neq 2$, then let i be the smallest exceedant of δ, with $i \neq 1$ (if such an integer i exists). The derangement $\delta' = \phi(\delta)$ is obtained by multiplying the derangement δ on the right by the transposition $(i-1\ i)$. In other terms, δ' is obtained from δ by exchanging the integers i and $i-1$ in the word $\delta(1)\cdots\delta(n)$.
   For example, if we take the derangement $\delta = 9\ 5\ 4\ 7\ 1\ 8\ 2\ 3$, then the smallest exceedant is the integer 2 and the derangement δ' is equal to 9 5 4 7 2 8 1 3.
   (b) If $\delta(n) = 2$, then let i be the smallest exceedant of δ, with $i > 2$ such that $\delta(i-1) \neq i$ (if such an integer exists). The derangement $\delta' = \phi(\delta)$ is obtained by multiplying the derangement δ on the right by the transposition $(i-1\ i)$. In other terms, δ' is obtained from δ by exchanging the integers i and $i-1$ in the word $\delta(1)\cdots\delta(n)$.
   For example, if we take the derangement $\delta = 9\ 4\ 1\ 6\ 7\ 3\ 8\ 2$, then the integer i is equal to 4 and the derangement δ' is equal to 9 3 1 6 7 4 8 2.

Definition 5.1. *Consider the following derangements for a given integer n:*

$$\gamma_{n,1} = (n\ n-1 \cdots\ 3\ 2\ 1),$$
$$\gamma_{n,i} = (n\ 2\ 3\ \cdots\ i-1\ i\ n-1\ n-2 \cdots i+1\ 1) \ \textit{for}\ \ i = 2, \cdots, n-2,$$
$$\gamma_{n,n-1} = (1\ 2\ 3 \cdots n-1\ n).$$

We will call these "critical derangements" and denote by C_n the set of the derangements $\gamma_{n,i}$ for all integer $i = 1, \cdots, n-1$.

Example 5.2. For $n = 6$, we have:

$$\gamma_{6,1} = (6\ 5\ 4\ 3\ 2\ 1)$$
$$\gamma_{6,2} = (6\ 2\ 5\ 4\ 3\ 1)$$
$$\gamma_{6,3} = (6\ 2\ 3\ 5\ 4\ 1)$$
$$\gamma_{6,4} = (6\ 2\ 3\ 4\ 5\ 1)$$
$$\gamma_{6,5} = (1\ 2\ 3\ 4\ 5\ 6).$$

Remark 5.3. *For all integer k in the set $[n-1]$, the number of exceedances of the derangement $\gamma_{n,k}$ is equal to k.*

Proposition 5.4. *The map ϕ preserves the property of being a derangement. In other terms, for all derangement δ in the set D_n, the permutation $\phi(\delta)$ is a derangement as well.*

Proof. Let us look at the various cases.

(1) If $\delta(n) \neq 1$ and $\delta(1) \neq n$, the exchange of the two integers $\delta(1)$ and $\delta(n)$ does not create any fixed point.

(2) If $\delta(n) = 1$, we exchange the integers i and $i+1$, where i is the largest anti-exceedant smaller than the integer n. This exchange does not create any fixed point. Indeed, $\delta'(i) \neq i$ because otherwise, $\delta(i) = i+1$ and i would not be an anti-exceedant. Also, $\delta'(i+1) \neq i+1$ because otherwise $\delta(i+1) = i$ and hence $i+1$, and not i would be the largest anti-exceedant.

(3) If $\delta(1) = n$ and $\delta(n) \neq 1$, we exchange the integers i and $i-1$ where i is either the smallest exceedant greater than 1 or the smallest anti-exceedant greater than the integer 2 and such that $\delta(i-1) \neq i$. This exchange does not create create any fixed point. Indeed, if we had $\delta'(i) = i$ then $\delta(i-1) = i$, which contradicts the definition of i in both subcases. If we had $\delta'(i-1) = i$ then we would have $\delta(i) = i-1$, which contradicts the fact that i is an exceedant.

□

Proposition 5.5. *The map ϕ is not defined on the elements of the set C_n.*

Proof. The critical derangement $\gamma_{n,1}$ falls in the case 3a of the definition of ϕ, the derangements $\gamma_{n,2}, \cdots, \gamma_{n,n-2}$ fall in the case 3b of the same definition and $\gamma_{n,n-1}$ in the case 2. In each of the cases, these are respectively the only derangements for which ϕ is not defined, because there does not exist an integer i having the property respectively described in each of the cases. □

Proposition 5.6. *The map ϕ changes the parity of a given derangement, that is, if δ is an even derangement then $\phi(\delta)$ is an odd derangement and vice-versa.*

Proof. The action of ϕ consists in multiplying a derangement by a transposition. This operation changes the parity. □

Proposition 5.7. *The map ϕ preserves the set of exceedances of all derangements, that is, the derangement δ has an exceedance in an integer j of the set $[n-1]$ if and only if $\phi(\delta)$ has an exceedance in j.*

Proof. If δ is a derangement such that $\delta(1) \neq n$ and $\delta(n) \neq 1$, then the exchange of the images of 1 and n does not change the fact that we have an exceedance in 1 and an anti-exceedance in n.

In all other cases, we exchange two consecutive integers, say i and $i+1$, as images of two other integers $\delta^{-1}(i)$ and $\delta^{-1}(i+1)$. Since both δ and $\delta' = \phi(\delta)$ are derangements, we have :

$$\delta^{-1}(i) < i = \delta(\delta^{-1}(i)) \Longleftrightarrow \delta^{-1}(i) < i+1 = \delta'(\delta^{-1}(i))$$

and

$$\delta^{-1}(i+1) < i+1 = \delta(\delta^{-1}(i+1)) \Longleftrightarrow \delta^{-1}(i) < i = \delta'(\delta^{-1}(i+1))$$

so, δ has an exceedance in $\delta^{-1}(i)$ (respectively, in $\delta^{-1}(i+1)$) if and only if δ' has an exceedance in $\delta^{-1}(i)$ (respectively, in $\delta^{-1}(i+1)$).

In either case, the set of exceedances of δ' is the same as the set of exceedances of δ. □

Corollary 5.8. *The map ϕ preserves the number of exceedances of a given derangement, that is, if δ is a derangement with k exceedances, then $\phi(\delta)$ has also k exceedances.*

Theorem 5.9. *The map ϕ is a bijection on the set $D_n \setminus C_n$ onto itself.*

Proof. Notice that if ϕ is defined on a derangement δ, then ϕ is also defined on $\delta' = \phi(\delta)$, because it is impossible to obtain a critical derangement as image of another derangement. Notice also that if δ is a derangement belonging to one of the cases 1, 2, or 3, respectively, then $\delta' = \phi(\delta)$ belongs to the same case ; moreover, the two integers that need to be exchanged in δ' to compute $\phi(\delta') = \phi(\phi(\delta))$ from δ' are the same as the two integers that need to be exchanged in δ to compute $\delta' = \phi(\delta)$ from δ.

Therefore, if σ is the transposition such that $\phi(\delta) = \delta\sigma$ then :

$$\phi(\phi(\delta)) = \phi(\delta\sigma) = \delta\sigma\sigma = \delta.$$

Hence, ϕ is an involution and therefore it is a bijection. □

Corollary 5.10. *The map ϕ is a bijection of the set $D_{n,k} \setminus \{\gamma_{n,k}\}$ onto itself.*

Corollary 5.11. *For all positive integers n and $1 \leq k \leq n-1$ one has:*

$$i_{n,k} - p_{n,k} = (-1)^n.$$

Proof. For all integer $k = 1, \cdots, n-1$, there exists a unique derangement $\gamma_{n,k}$ of the set $D_{n,k}$ which is an element of the set C_n. Furthermore, this derangement $\gamma_{n,k}$ is always a cycle of length n and hence $\gamma_{n,k}$ is an even pernutation if and only if the integer n is odd and vice-versa. Therefore, if δ is a derangement of the set $D_{n,k}$ and is not an element of C_n, the map ϕ associates bijectively to this derangement δ a derangement δ' in the set $D_{n,k}$ having the opposite parity. Hence, the numbers $i_{n,k}$ and $p_{n,k}$ differ only by 1 and the sign of this difference is given by the parity of the derangement $\gamma_{n,k}$. □

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E-mail address: `mantaci@liafa.jussieu.fr, frakoton@syfed.refer.mg`

LIAFA, Université Paris 7 - Denis Diderot and, Département de Mathématiques, Faculté des Sciences, Université d'Antananarivo