DEFORMATIONS OF WEYL'S DENOMINATOR FORMULA FOR $sp(2n)$

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ABSTRACT. Tokuyama has given a formula that is a deformation of the Weyl denominator formula for $sl(n)$. This formula can also be considered as a generalization of the ratio of alternants formula for Schur functions. Here we derive a symplectic version of Tokuyama's result, namely a deformation of Weyl's denominator formula for $sp(2n)$. We further show how it specializes to give symplectic versions of a number of classical identities.

RÉSUMÉ. Tokuyama a dérivé une déformation pour $sl(n)$ du formule dénominateur de Weyl. Cette formule est une généralisation de la formule du taux d'alternants pour les fonctions de Schur. Ici, nous dérivons une version symplectique du résultat de Tokuyama. Ceci est une déformation pour $sp(2n)$ de la formule du dénominateur de Weyl. Nous donnons aussi des versions symplectiques de plusieurs identités classiques.

1. INTRODUCTION

Weyl's denominator formula [14] is a well–known algebraic identity that expresses a certain product taken over the positive roots of a semisimple Lie algebra as a sum taken over the elements of the corresponding Weyl group of the Lie algebra. For $A_{n-1} = sl(n)$ or $gl(n)$, introducing variables $x_1, x_2, \ldots x_n$ in place of basis elements gives rise to a symmetric function identity with a combinatorial interpretation [1]; essentially, the ratio of alternants form for Schur functions. Introducing a parameter t for this same Lie algebra, $gl(n)$, produces a further identity, a so–called deformation of Weyl's denominator formula [13]. This particular deformation was inspired in part by the work of Mills, Robbins, and Rumsey [5] which has achieved prominence because of its association with alternating sign matrices.

Deformations for other classical Lie algebras, $B_n = so(2n+1), C_n = sp(2n)$ and $D_n = so(2n)$ have also been derived [6, 8, 9]. Proof techniques for these results have been both algebraic and combinatorial (and a combination of the two). In particular, Tokuyama's proof of the original deformation used representation theoretic methods, but the result was later proved combinatorially by Okada [6]. In this work we investigate the most natural deformation of Weyl's denominator formula in the case of $C_n = sp(2n)$ and derive a direct generalization of Tokuyama's result. Our proof uses extensions of Okada's methods. We also discuss a number of specializations that demonstrate connections to the work of Weyl and Mills, Robbins, and Rumsey.

2. YOUNG DIAGRAMS AND DEFORMATIONS OF WEYL'S DENOMINATOR FORMULA

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition, that is a weakly decreasing sequence of non-negative integers λ_i. The weight, $|\lambda|$, of the partition λ is the sum of its parts, and its length, $\ell(\lambda) \leq n$, is the number of its non-zero parts. Each such partition λ defines a Young diagram F^λ consisting of $|\lambda|$ boxes arranged in $\ell(\lambda)$ rows of lengths λ_i that are left adjusted to a vertical line. Formally, $F^\lambda = \{(i,j) \,|\, 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$.

Recalling that δ is the partition $\delta = (n, n-1, \ldots, 1)$, then $\mu = \lambda + \delta$ is a partition all of whose parts $\mu_i = \lambda_i + n - i + 1$ are distinct and non-zero. Thus μ is a strongly decreasing sequence of n positive integers. More generally, any partition μ all of whose parts are distinct, defines a shifted Young diagram SF^μ consisting of $|\mu|$ boxes arranged in $\ell(\mu)$ rows of lengths μ_i that are left adjusted to a diagonal line. To be precise, $SF^\mu = \{(i,j) \,|\, 1 \leq i \leq \ell(\mu), i \leq j \leq \mu_i + i - 1\}$.

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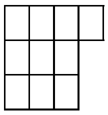
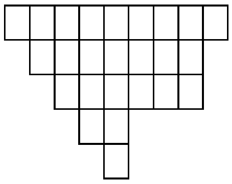
For example, $\lambda=(4,3,3)$ and $\mu=(9,7,6,2,1)$ we have

(2.1) $F^\lambda =$  and $SF^\mu =$ 

We can use these diagrams to generate symmetric functions. In the case of F^λ we produce first the set of $sl(n)$-standard tableaux, $\mathbf{T}^\lambda(n)$, that results from filling F^λ with integers from 1 to n with the following restrictions:

- entries weakly increase across rows
- entries strictly increase down columns

Taking a weight of x_i for each i that appears in a tableau and summing over all tableaux of shape λ we obtain the Schur function of shape λ:

$$s_\lambda(x) = s_\lambda(x_1,x_2,\ldots,x_n) = \sum_{T\in\mathbf{T}^\lambda(n)} x^{\mathrm{wgt}(T)}, \tag{2.2}$$

where the sum is over all $sl(n)$-standard tableaux T of shape λ and

$$x^{\mathrm{wgt}(T)} = x_1^{m_1(T)} x_2^{m_2(T)} \cdots x_n^{m_n(T)} \tag{2.3}$$

with $m_k(T)$ equal to the number of entries k in T [4] [10].

Note also that the Schur functions have a ratio of alternants form as well:

$$s_\lambda(x) = \frac{|x_i^{\lambda_j+n-j+1}|}{|x_i^{n-j+1}|},$$

where the denominator can be recognized as the Vandermonde determinant.

The symplectic character, $sp(2n)$, also has a definition in terms of tableaux [2, 3, 12], specifically $sp(2n)$-standard tableaux. $Sp(2n)$-standard tableaux, $\mathbf{T}^\lambda(n,\overline{n})$, result from filling F^λ with integers from 1 to n and $\overline{1}$ to $\overline{n}$ ordered $\overline{1}<1<\overline{2}<2<\ldots<\overline{n}<n$ subject to the following restrictions:

- entries weakly increase across rows
- entries strictly increase down columns
- the first entry in row k is no greater than $\overline{k}$ for $k=1,2,\ldots,n$.

Taking a weight of x_i or $x_{\overline{i}} = x_i^{-1}$ for each i or $\overline{i}$ that appears in a tableau and summing over all tableaux of shape λ we obtain the symplectic character for λ:

$$sp_\lambda(x) = \sum_{T\in\mathbf{T}^\lambda(n,\overline{n})} x^{\mathrm{wgt}(T)}, \tag{2.4}$$

where the sum is over all $sp(2n)$-standard tableaux T of shape λ and

$$x^{\mathrm{wgt}(T)} = x_1^{m_1(T)-m_{\overline{1}}(T)} x_2^{m_2(T)-m_{\overline{2}}(T)} \ldots x_n^{m_n(T)-m_{\overline{n}}(T)}.$$

Thus each entry k or $\overline{k}$ contributes a factor x_k or $x_{\overline{k}}$ to $x^{\mathrm{wgt}(S)}$.

By way of illustration, in the case $n=5$, $\lambda=(4,3,3)$ and $\mu=(9,7,6,2,1)$ we have typically:

$$T = \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\\hline 2&3&3\\\cline{1-3} 4&4&5\\\cline{1-3}\end{array} \in \mathbf{T}^{433}(5) \quad , \quad T = \begin{array}{|c|c|c|c|}\hline \overline{1}&1&\overline{2}&4\\\hline 3&\overline{4}&\overline{4}\\\cline{1-3} \overline{4}&4&4\\\cline{1-3}\end{array} \in \mathbf{T}^{433}(5,\overline{5}) \tag{2.5}$$

In the case of SF^μ we define two different sets of tableaux, one corresponding to the known result of Tokuyama (the $sl(n)$-standard shifted tableaux) and one corresponding to our result (the $sp(2n)$-standard shifted tableaux).

The first set, the $sl(n)$-standard shifted tableaux, $\mathbf{ST}^\mu(n)$, results from filling SF^μ with integers from 1 to n subject to the following restrictions:

- entries must weakly increase across rows and down columns
- entries must strictly increase along top-left-to-bottom-right diagonals

The second set, the $sp(2n)$-standard shifted tableaux, $\mathbf{ST}^{\mu}(n,\overline{n})$, results from filling SF^{μ} with integers from 1 to n and $\overline{1}$ to $\overline{n}$ ordered $\overline{1} < 1 < \overline{2} < 2 < \ldots < \overline{n} < n$ subject to the two restrictions given above and the additional restriction:

- the first entry in row k must be either k or $\overline{k}$ for $k = 1, 2, \ldots, n$.

Typically, for $n = 5$, $\lambda = (4,3,3)$ and $\mu = (9,7,6,2,1)$ we have

$$S = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1&1&1&2&2&3&3&4&5 \\ \hline \multicolumn{1}{c|}{} &2&2&2&3&4&4&4& \multicolumn{1}{|c}{} \\ \cline{2-8} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} &3&4&5&5&5&5& \multicolumn{1}{|c}{} \\ \cline{3-8} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} &4&5& \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{4-5} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} &5& \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{5-5} \end{array} \in \mathbf{ST}^{97621}(5), \quad S = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \overline{1}&1&\overline{2}&2&\overline{3}&\overline{3}&\overline{4}&4&5 \\ \hline \multicolumn{1}{c|}{} &\overline{2}&\overline{2}&2&3&\overline{4}&\overline{4}&4& \multicolumn{1}{|c}{} \\ \cline{2-8} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} &3&\overline{4}&4&4&4&4& \multicolumn{1}{|c}{} \\ \cline{3-8} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} &4&4& \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{4-5} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} &\overline{5}& \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{5-5} \end{array} \in \mathbf{ST}^{97621}(5,\overline{5}) \tag{2.6}$$

We make the following additional definitions:

Definition *The ribbon strip,* $\mathrm{str}_k(S)$ *(respectively* $\mathrm{str}_{\overline{k}}(S)$*) consists of all boxes in the* $sl(n)$*- (resp.* $sp(2n)$*-)standard shifted tableau containing* k *(resp.* $\overline{k}$*) with no two such boxes on the same diagonal.*

Definition *The number of boxes in* $\mathrm{str}_k(S)$ *(resp.* $\mathrm{str}_{\overline{k}}(S)$*) is denoted* $\mathrm{wgt}_k(S)$ *(resp.* $\mathrm{wgt}_{\overline{k}}(S)$*).*

Definition *The number of rows contained in* $\mathrm{str}_k(S)$ *(resp.* $\mathrm{str}_{\overline{k}}(S)$*) is denoted* $\mathrm{row}_k(S)$ *(resp.* $\mathrm{row}_{\overline{k}}(S)$*).*

Definition *The number of connected components of* $\mathrm{str}_k(S)$ *(resp.* $\mathrm{str}_{\overline{k}}(S)$*) is denoted* $\mathrm{con}_k(S)$ *(resp.* $\mathrm{con}_{\overline{k}}(S)$*).*

Then make the following definitions for shifted tableaux. In the $sl(n)$ case:

$$\begin{aligned} \mathrm{str}(S) &= \sum_{k=1}^{n} \mathrm{con}_k(S) \\ \mathrm{hgt}(S) &= \sum_{k=1}^{n} (\mathrm{row}_k(S) - \mathrm{con}_k(S)) \\ x^{\mathrm{wgt}(S)} &= \prod_{k=1}^{n} x_k^{\mathrm{wgt}_k(S)} \end{aligned}$$

In the $sp(2n)$ case:

$$\begin{aligned} \mathrm{str}(S) &= \sum_{k=1}^{n} (\mathrm{con}_k(S) + \mathrm{con}_{\overline{k}}(S)) \\ \mathrm{hgt}(S) &= \sum_{k=1}^{n} (\mathrm{row}_k(S) - \mathrm{con}_k(S) - \mathrm{row}_{\overline{k}}(S)) \\ x^{\mathrm{wgt}(S)} &= \prod_{k=1}^{n} x_k^{\mathrm{wgt}_k(S) - \mathrm{wgt}_{\overline{k}}(S)} \\ \mathrm{bar}(S) &= \sum_{k=1}^{n} \mathrm{wgt}_{\overline{k}}(S) \end{aligned}$$

Now we turn to Weyl's denominator formula and its deformations. For any simple Lie algebra g of a Lie group G, Weyl's denominator formula [14] takes the form:

$$e^{\delta} \prod_{\alpha \in \Delta_+} \left(1 - e^{-\alpha}\right) = \sum_{w \in W} \mathrm{sgn}(w) e^{w\delta}, \tag{2.7}$$

where the product on the left is over all α in the set, Δ_+, of positive roots of g and the sum on the right is over all elements w of the Weyl group, W, of g. The symbol δ is half the sum of the positive roots and $\mathrm{sgn}(w) = (-1)^{\ell(w)}$ where $\ell(w)$ is the length of w when expressed as a word in the generators of W.

This formula (2.7) can be specialized for particular Lie algebras. For the Lie algebra $A_{n-1} = sl(n)$ of the Lie group $SL(n)$, (2.7) becomes

$$D_{sl(n)}(x) = \prod_{1\le i\le n} x_i^{n-i+1} \prod_{1\le i<j\le n} (1 - x_i^{-1}x_j) = \sum_{\pi\in S_n} sgn(\pi) x_{\pi_1}^n x_{\pi_2}^{n-1} \dots x_{\pi_n}^1 = |x_i^{n-j+1}| \tag{2.8}$$

which is the Vandermonde determinant. This follows since

$$\Delta_+ = \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n\}, \tag{2.9}$$

with $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n = 0$, and thus $\delta = n\epsilon_1 + (n-1)\epsilon_2 + \cdots + \epsilon_n$. The corresponding Weyl group is $W = S_n$, the symmetric group, and each $w = \pi \in S_n$ maps ϵ_i to ϵ_{π_i} for $i = 1, 2, \dots, n$. To get the expression in (2.9), set $x_i = e^{\epsilon_i}$ for $i = 1, 2, \dots, n$ and $x = (x_1, x_2, \dots, x_n)$.

For the Lie algebra $C_n = sp(2n)$ of the Lie group $Sp(2n)$, (2.7) becomes

$$D_{sp(2n)}(x) = \prod_{1\le i\le n} x_i^{n-i+1} \prod_{1\le i\le n} (1 + tx_i^{-2}) \prod_{1\le i<j\le n} (1 - x_i^{-1}x_j)(1 - x_i^{-1}x_j^{-1}). \tag{2.10}$$

This follows since

$$\Delta_+ = \{2\epsilon_i \mid 1 \le i \le n\} \cup \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\}. \tag{2.11}$$

and thus $\delta = n\epsilon_1 + (n-1)\epsilon_2 + \cdots + \epsilon_n$. The corresponding Weyl group is $W = H_n = S_2 \wr S_n$, the hyperoctohedral group, and each $w = \tilde{\pi} \in H_n$ maps ϵ_i to $\pm\epsilon_{\pi_i}$ for $i = 1, 2, \dots, n$. To get the expression in (2.10), set $x_i = e^{\epsilon_i}$ for $i = 1, 2, \dots, n$.

Each finite-dimensional irreducible representation of $sl(n)$ is specified by a highest weight vector λ, which in the ϵ-basis takes the form $\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_n\epsilon_n$ with λ_i an integer for $i = 1, 2, \dots, n$ and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$. Equivalently, we may specify this irreducible representation by the corresponding partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and its character is given by the Schur function defined above. Similarly, each finite-dimensional irreducible representation of $sp(2n)$ is specified by its highest weight vector λ which in the ϵ-basis again takes the form $\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_n\epsilon_n$ with λ_i an integer for $i = 1, 2, \dots, n$ and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$. Equivalently, we may again specify the irreducible representation by the corresponding partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and its character $sp_\lambda(x)$ may be defined in terms of tableaux as above.

The object of this paper is to study deformations of Weyl's denominator formula, (2.7). Weyl's formula can be deformed by the introduction of a parameter t which replaces the -1 in the left hand side of (2.7):

$$D_g(t) = e^\delta \prod_{\alpha\in\Delta_+} \left(1 + te^{-\alpha}\right). \tag{2.12}$$

For the Lie algebras A_{n-1} and C_n, we can similarly deform to obtain

$$D_{sl(n)}(x;t) = \prod_{1\le i\le n} x_i^{n-i+1} \prod_{1\le i<j\le n} (1 + tx_i^{-1}x_j), \tag{2.13}$$

and

$$D_{sp(2n)}(x;t) = \prod_{1\le i\le n} x_i^{n-i+1} \prod_{1\le i\le n} (1 + tx_i^{-2}) \prod_{1\le i<j\le n} (1 + tx_i^{-1}x_j)(1 + tx_i^{-1}x_j^{-1}). \tag{2.14}$$

Setting $t=-1$ returns (2.9) and (2.10) above.

Our main result is a symplectic version of a further deformation due to Tokuyama [13]. Tokuyama's result is an expansion of the deformed product $D_{sl(n)}(x;t)s_\lambda$. Our result, introduced in Theorem 2 below, is an expansion of the deformed symplectic product $D_{sp(2n)}(x;t)sp_\lambda(x;t)$.

For our purposes, it is convenient in the case of $sp(2n)$ to deform not just the denominator, as in (2.14), but also the character (2.4) by allowing each entry $\overline{k}$ to contribute not just a factor $x_{\overline{k}}=x_k^{-1}$ but $t^2x_{\overline{k}}$. This leads to the definition

$$sp_\lambda(x;t)=\sum_{T\in\mathbf{T}^\lambda(n,\overline{n})} t^{2\mathrm{bar}(T)}\ x^{\mathrm{wgt}(T)}, \tag{2.15}$$

where recall bar(T) is the number of barred entries in T, that is

$$\mathrm{bar}(T)=\sum_{k=1}^{n} m_{\overline{k}}(T). \tag{2.16}$$

We have the following theorems:

Theorem 1 (Tokuyama, 1988). *Let λ be a partition of length $\ell(\lambda)\le n$ and let $\delta=(n,n-1,\dots,1)$. Then*

$$D_{sl(n)}(x;t)\ s_\lambda(x)=\sum_{S\in\mathbf{ST}^{\lambda+\delta}(n)} t^{\mathrm{hgt}(S)}\ (1+t)^{\mathrm{str}(S)-n}\ x^{\mathrm{wgt}(S)} \tag{2.17}$$

To view this as a generalization of the ratio of alternants form for Schur functions, recall that for $t=-1$, $D_{sl(n)}(x;t)$ reduces to $D_{sl(n)}(x)$ which in turn is the Vandermonde determinant (see (2.9)), and note that the right hand side disappears unless $\mathrm{str}(S)=n$ in which case the right hand side becomes $\sum_{\pi\in S_n} sgn(\pi)x_{\pi_1}^{\lambda_1+n}x_{\pi_2}^{\lambda_2+n-1}\dots x_{\pi_n}^{\lambda_n+1}=|x_i^{\lambda_j+n-j+1}|$.

The main theorem of the paper is:

Theorem 2. *Let λ be a partition of length $\ell(\lambda)\le n$ and let $\delta=(n,n-1,\dots,1)$. Then*

$$\begin{aligned} &D_{sp(2n)}(x;t)\ sp_\lambda(x;t)\\ &=\sum_{S\in\mathbf{ST}^{\lambda+\delta}(n,\overline{n})} t^{\mathrm{hgt}(S)+2\mathrm{bar}(S)}\ (1+t)^{\mathrm{str}(S)-n}\ x^{\mathrm{wgt}(S)}\end{aligned} \tag{2.18}$$

Tokuyama's theorem has been proved by Okada [6] using a determinantal expression and lattice paths. This same approach can be modified to prove Theorem 2.

3. Specializations

Theorem 2 is rich in corollaries which arise by specializing, both in turn and in various combinations, the parameter t, the partition λ and the indeterminates $x=(x_1,x_2,\dots,x_n)$.

First of all, the case $t=-1$ and allowing λ to be any fixed partition of length $\ell(\lambda)\le n$, we obtain Weyl's character formula for $sp(2n)$ from Theorem 2.

Corollary 3 (Weyl, 1926). *Let W be the Weyl group of $sp(2n)$, let δ be half the sum of the positive roots of $sp(2n)$ and let λ be an arbitrary partition of length $\ell(\lambda)\le n$. Then the irreducible representation of $sp(2n)$ specified by λ has character:*

$$sp_\lambda(x)=\sum_{w\in W}\mathrm{sgn}(w)e^{w(\lambda+\delta)}\Big/\sum_{w\in W}\mathrm{sgn}(w)e^{w(\delta)}. \tag{3.19}$$

From the definition (2.12), setting $t=-1$ in $D_{sp(2n)}(x;t)$ gives the undeformed Weyl denominator function for $sp(2n)$. It follows that

$$D_{sp(2n)}(x;-1) = \sum_{w\in W} \mathrm{sgn}(w) e^{w(\delta)} \tag{3.20}$$

is just the original denominator formula of Weyl [14] for the case of the Lie algebra $sp(2n)$:

Corollary 4 (Weyl, 1926). *Let Δ_+ be the set of positive roots of $sp(2n)$, let δ be half the sum of these positive roots, and let W be the Weyl group of $sp(2n)$. Then*

$$e^{\delta} \prod_{\alpha\in\Delta_+} (1-e^{-\alpha}) = \sum_{w\in W} \mathrm{sgn}(w)\ e^{w(\delta)}. \tag{3.21}$$

It might be thought that the case $t=0$ of Theorem 2 might provide something new. This is not the case however, since it leads only to the confirmation of the fact that

$$sp_\lambda(x;0) = s_\lambda(x). \tag{3.22}$$

Turning to the more interesting case of $t=1$, setting $t=1$ in Theorem 2 leads directly to:

Corollary 5. *Let λ be a partition into no more than n parts and let δ be the partition $(n,n-1,\ldots,1)$, then*

$$D_{sp(2n)}(x;1)\ sp_\lambda(x) = \sum_{S\in\mathbf{ST}^{\lambda+\delta}(n,\overline{n})} 2^{\mathrm{str}(S)-n}\ x^{\mathrm{wgt}(S)}. \tag{3.23}$$

This represents what might be called a symplectic version of the formula given by Stanley [11] for the $t=-1$ case of the Hall-Littlewood function $P_\mu(x;t)$ with $\mu=\lambda+\delta$.

Making the further specialisation $x_i=1$ for all $i=1,2,\ldots,n$ in Corollary 4.3, and using (2.14) to note that under the same specialisation $D_{sp(2n)}(x;1)$ is just $2^{n(n+1)/2}$, leads to:

Corollary 6. *Let λ be a partition into no more than n parts and let δ be the partition $(n,n-1,\ldots,1)$, then*

$$2^{\frac{1}{2}n(n+1)}\,\mathrm{dim}_{2n}(sp_\lambda) = \sum_{S\in\mathbf{ST}^{\lambda+\delta}(n,\overline{n})} 2^{\mathrm{str}(S)-n}, \tag{3.24}$$

where $\mathrm{dim}_{2n}(sp_\lambda) = sp_\lambda(1,1,\ldots,1)$ *is the dimension of the irreducible representation of $sp(2n)$ having character $sp_\lambda(x)$.*

This, in turn, is the symplectic version of a formula due to Mills, Robbins and Rumsey [5]. The relevant formula is that of their Theorem 2 which involves a factor that can be identified as nothing other than $\mathrm{dim}_n(sl_\lambda)$ where $\lambda_i = a_{n-i+1}-n+i-1$ for $i=1,2,\ldots,n$.

Alternatively, making the further specialisation $\lambda=0$ in Corollary 4.3, and employing (2.14) in Theorem 1.2, leads to:

Corollary 7. *Let $\delta=(n,n-1,\ldots,1)$. Then*

$$\begin{aligned}\prod_{1\le i\le n} x_i^{n-i+1} \prod_{1\le i\le n} (1+x_i^{-2}) \prod_{1\le i<j\le n} &(1+x_i^{-1}x_j)(1+x_i^{-1}x_j^{-1}) \\ &= \sum_{S\in\mathbf{ST}^{\delta}(n,\overline{n})} 2^{\mathrm{str}(S)-n}\ x^{\mathrm{wgt}(S)}.\end{aligned} \tag{3.25}$$

Generalising this for all t, if one sets $\lambda=0$ in Theorem 2 and replaces x_i by its inverse x_i^{-1} for all $i=1,2,\ldots,n$, then one obtains an identity which is reminiscent of, but somewhat simpler than, the deformations of Weyl's denominator formulae due to Okada [7] and Simpson [8, 9]:

Corollary 8. *Let $\delta = (n, n-1, \ldots, 1)$. Then*

$$\begin{aligned}
&\prod_{1\le i\le n} (1+tx_i^2) \prod_{1\le i<j\le n} (1+tx_ix_j)(1+tx_ix_j^{-1}) \\
(3.26)\qquad &= \sum_{S\in\mathbf{ST}^\delta(n,\overline{n})} t^{\mathrm{hgt}(S)+2\mathrm{bar}(S)}\,(1+t)^{\mathrm{str}(S)-n}\,x^{\delta-\mathrm{wgt}(S)},
\end{aligned}$$

where $x^{\delta-\mathrm{wgt}(S)} = x^\delta x^{-\mathrm{wgt}(S)}$ with $x^\delta = x_1^n x_2^{n-1}\ldots x_n$.

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References

[1] Gessel, I., "Tournaments and Vandermonde's determinant." *J Graph Theory* **3** (1979), 305-307.
[2] King, R.C., "Weight multiplicities for the classical Lie groups." In *Lecture Notes in Physics* **50** Springer, New York, 1976, 490–499.
[3] King, R.C., and El-Sharkaway, N.G.I., "Standard Young tableaux and weight multiplicities of the classical Lie groups." *J Phys A* **16** (1983), 3153–3177.
[4] Macdonald, I.G., *Symmetric Functions and Hall Polynomials.* Oxford University Press, Oxford, 1979.
[5] Mills, W.H., Robbins, D.P., and Rumsey Jr, H., "Alternating sign matrices and descending plane partitions." *J Combin Theory Ser A* **34** (1983), 340–359.
[6] Okada, S. "Partially strict shifted plane partitions." *J Combin Theory Ser A* **53** (1990), 143–156.
[7] Okada, S., "Alternating sign matrices and and some deformations of of Weyl's denominator formula." *J Alg Combin* **2** (1993), 155–176.
[8] Simpson, T. "Three generalizations of Weyl's denominator formula." *Electronic J Combin* **3** (1997), #R12.
[9] Simpson, T., "Another deformation of Weyl's denominator formula." *J Combin Theory Ser A* **77** (1997), 349–356.
[10] Stanley, R.P., "Theory and applications of plane partitions I." *Stud App Math* **50** (1971), 167–188.
[11] Stanley, R.P., "Problem Session: Problem 4." In *Combinatorics and Algebra* Ed. C Greene, *Contemporary Maths* **34**, Amer Math Soc, Providence, RI, 1984, 304–305.
[12] Sundaram, S., "Tableaux in the representation theory of the clasical Lie groups." in *Invariant Theory and Tableaux* Ed. D Stanton, **IMA Vol 19**, Springer-Verlag, New York, 1989, 191–225.
[13] Tokuyama, T., "A generating function of strict Gelfand patterns and some formulas on characters of general linear groups." *J Math Soc Japan* **40** (1988), 671–685.
[14] Weyl, H., "Theorie der Darstellungen kontinuierlicher halb-einfacher Gruppen durch lineare Transformationen III." *Math Zeit* **24** (1926), 377–396.

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