# PSEUDO-PERMUTATIONS I: FIRST COMBINATORIAL AND LATTICE PROPERTIES

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ABSTRACT. In this paper, we introduce new combinatorial objects, the pseudo-permutations, which are a generalization of the permutations. Pseudo-permutations naturally appear in various fields of Computer Science and Mathematics. We provide the first combinatorial results on these objects: we study the classical statistics of enumeration, inversions, descents and we prove that the set of all pseudo-permutations has a lattice structure.

RÉSUMÉ. Cet article présente une étude de nouveaux objets combinatoires baptisés les pseudo-permutations, qui sont des généralisations des permutations. Les pseudo-permutations sont apparues naturellement dans divers cadres informatiques et mathématiques. Nous présentons ici une première étude de ces objets, en calculant les statistiques énumératives classiques, comme les inversions et les descentes et nous montrons que l'ensemble des pseudo-permutations est un treillis.

**Keywords:** Permutations, Enumerative Combinatorics, Lattices

## 1. Introduction

In various fields of Computer Science (Artificial Intelligence, Knowledge representation, ...), one has to deal with temporal knowledge (see, *e.g.*, [All81, Yu83]: one considers a set of events which happen at certain dates, and wants to use this information to solve a problem, take a decision. However, it is often not meaningful *when* the events occur, while the relevant informations are the *temporal relations* between events: did event $i$ happened before, during or after event $j$ ? In this context, it is natural to represent the temporal relations between $n$ events by an ordered sequence of nonempty parts of $[\![1,n]\!]$ such that each integer appears exactly once. If $i$ is in a part of $[\![1,n]\!]$ which appears in the sequence before the part which contains $j$, then the event $i$ happened before the event $j$. If they appear in the same part, they occured simultaneously. For example, the sequence $\{1\}\{3,4\}\{2\}$ means that event 1 occured first, event 2 occured last, and events 3 and 4 occured at the same time. In the following, we will call such a sequence a *pseudo-permutation of order* $n$ (in the example, $n=4$), we will use parentheses instead of braces, and we will remove unnecessary comas. Therefore, we will write the example as (1)(34)(2).

Having defined the pseudo-permutations, one can study their combinatorial properties. They appear to have a very rich structure and many interesting properties. Among these, one can see that their enumeration is related to Eulerian numbers, and that an inversion table associated with each pseudo-permutation can be defined. This induces a partial order on pseudo-permutations compatible with the inversion tables and we will see that this order is a lattice, as it is the case for the usual permutations. One can also define the descents of a pseudo-permutation and prove that this notion has the same properties as the usual descents on permutations. About the lattice structure, one can also see that the set can be divided in connected components that are hypercubes, the dimensions of which can naturally be interpretated in combinatorial terms. The aim of this paper is to show how rich can be this structure and to give a first insight in it, given that a second paper is in preparation to provide new interesting properties about pseudo-permutations. Since this paper is intented to present this new object, we will only give the sketches of the proofs and will publish a long version elsewhere.

The paper is structured as follows. We first recall the definitions of a lattice and the usual properties of Eulerian numbers (Section 2). We then concentrate on the pseudo-permutations themselves and define the pseudo-permutohedron (Section 3). Next, we prove their first combinatorial properties (Section 4) and their first lattice properties (Section 5).

## 2. Preliminaries

We recall here a few definitions and basic results used in the rest of the paper. We will also use the following standard notations. We will denote by $[\![ i,j ]\!]$ the set $\{k \in \mathbb{N} \mid i \leq k \leq j\}$ and by $\mathfrak{S}_n$ the set of all the permutations of order $n$, *i.e.*, the set of all the sequences of $n$ integers which contain each integer in $[\![ 1,n ]\!]$ exactly once. The *permutohedron* of order $n$ is then the directed graph defined over $\mathfrak{S}_n$ by asking that: there is an edge from $\sigma$ to $\sigma'$ if $\sigma'$ is obtained from $\sigma$ by switching two neighbour integers $i$ and $j$ in $\sigma$ such that $i < j$ (see Figure 1 for the permutohedron of order 3).

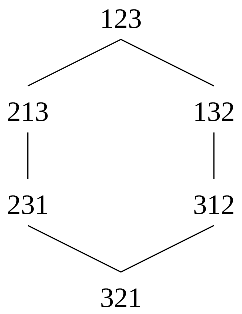


Figure 1. The permutohedron of order 3.

### 2.1. Rises and Eulerian numbers.

The Eulerian numbers are very classical numbers and the reader can refer to [FS70, Com70] for a complete overview about these. In this paper, we will only recall their usual definitions and properties.

Let $\sigma$ be a permutation. One says that $\sigma$ has a *rise* in $i$, or equivalently that $i$ is a *rise* of $\sigma$, if $\sigma(i) > \sigma(i+1)$. The *rise number* of $\sigma$ is then the number of rises of $\sigma$. The *rise set* of $\sigma$ is the set of the rises of $\sigma$. If one denotes by $a_{n,k}$ the number of elements of $\mathfrak{S}_n$ with $k$ rises, one has the following formulas:

$$a_{n,k} = a_{n,n-k-1} = (n-k)a_{n-1,k-1} + (k+1)a_{n-1,k},$$

with $n > 1$ and $k > 1$. Moreover, $a_{n,0} = 1$ and $a_{0,k} = 0$ for all $n$ and $k$. The *Eulerian numbers* are very close to these numbers since they can be defined by $A_{n,k} = a_{n,k-1}$. Moreover, the generating series of these numbers is known and one has:

$$1 + \sum_{1 \leq k \leq n} a_{n,k-1} \frac{t^n}{n} u^{k-1} = \frac{1-u}{e^{t(u-1)} - u}.$$

One can define in the same way the descents of a permutation and the descent number of a permutation (see, *e.g.*, [Com70] for more details).

### 2.2. Lattice theory.

We recall that an *order relation* is a binary relation $\leq$ over a set, such that for all $x$, $y$ and $z$ in this set, $x \leq x$ (reflexivity), $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity), and $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetry). Such a relation is often called a *partial order*. The set is then a *partially ordered set* or, for short, a *poset*.

A *lattice* is a poset such that any two elements $a$ and $b$ have a least upper bound (called *supremum* of $a$ and $b$ and denoted by $a \vee b$) and a greatest lower bound (called *infimum* of $a$ and $b$ and denoted by $a \wedge b$). The element $a \vee b$ is the smallest element among the elements greater than both $a$ and $b$. The element $a \wedge b$ is defined dually. Lattices are strongly structured sets, and many general results are known about them. For example, efficient coding and algorithms are known for lattices. For more details, see, *e.g.*, [DP90, Pha99].

## 3. Pseudo-permutations

3.1. **Definitions.** Let $n$ be an integer. The set $\mathfrak{P}(n)$ of the pseudo-permutations of order $n$ is the set of sequences of non-empty parentheses such that each integer in $[\![ 1, n ]\!]$ appears exactly once. In other words, the set $\mathfrak{P}(n)$ is the set of ordered partitions [1] of $[\![ 1, n ]\!]$ with nonempty parts.
For example, here is the complete set $\mathfrak{P}(3)$:

$$\begin{aligned}\mathfrak{P}(3) = \{ \quad & (1)(2)(3),\ (1)(3)(2),\ (2)(1)(3),\ (2)(3)(1),\ (3)(1)(2),\ (3)(2)(1), \\ & (1)(23),\ (2)(13),\ (3)(12),\ (23)(1),\ (13)(2),\ (12)(3),\ (123) \quad \}\end{aligned}$$

Since the order inside the parentheses is irrelevant, we will generally write the integers in the parentheses in increasing order. Notice also that there exists an obvious embedding from the permutations into the pseudo-permutations that maps a permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$ to $(\sigma_1)(\sigma_2)\cdots(\sigma_n)$.

As for the usual permutations, one can generate a graph which vertices are the elements of $\mathfrak{P}(n)$ and which edges are defined according to the following operators:

- The operator $M_i$ acts on the $i$-th and the $i+1$-th parentheses of a pseudo-permutation as follows: if each element of the $i$-th parenthese is smaller than all the elements of the $(i+1)$-th, then one can merge these two parentheses into one single parenthese which contains the union of the elements of these two parentheses.
- The operator $S_{i,j}$ acts on the $i$-th parenthese of a pseudo-permutation as follows: it splits this parenthese into two parentheses, the *second* one containing the $j$ smallest elements of the initial parenthese and the first one containing the others.

For example, one can apply the operators $M_2$, $M_4$, $S_{3,1}$, $S_{3,2}$ and $S_{4,1}$ to the permutation $\sigma = (7)(3)(568)(12)(4)$:

$$\begin{aligned}\sigma \quad & \xrightarrow{M_2} (7)(3568)(12)(4) \\ & \xrightarrow{M_4} (7)(3)(568)(124) \\ & \xrightarrow{S_{3,1}} (7)(3)(68)(5)(12)(4) \\ & \xrightarrow{S_{3,2}} (7)(3)(8)(56)(12)(4) \\ & \xrightarrow{S_{4,1}} (7)(3)(568)(2)(1)(4)\end{aligned}$$

We can now define the *pseudo-permutohedron*, denoted by $\mathfrak{G}(n)$, as the directed graph one obtains by iterating these two operators from the element $(1)(2)\cdots(n)$. For example, we give in Figure 2 the graphs $\mathfrak{G}(3)$ and $\mathfrak{G}(4)$.

This allows us to define a natural partial order on the pseudo-permutations as $\sigma < \tau$ if one can obtain $\sigma$ from $\tau$ by iterating operators $M$ and $S$. We will see in Section 4.2 that this order can also be defined with the notion of inversions and that it shows that $\mathfrak{G}(n)$ is a lattice.

We will now see some properties of the pseudo-permutations and their associated pseudo-permutohedron, first from the classical combinatorial point of view, then from the lattice point of view.

[1] A *partition* of a set $S$ is a set of subsets $S_1, S_2, \ldots, S_k$ of $S$ such that $\cup_{i=1}^{k} S_i = S$ and for all $i \neq j$, $S_i \cap S_j = \emptyset$. The $S_i$ are then the *parts* of the partition. A partition is *ordered* if we consider a sequence of subsets instead of a set of subsets.

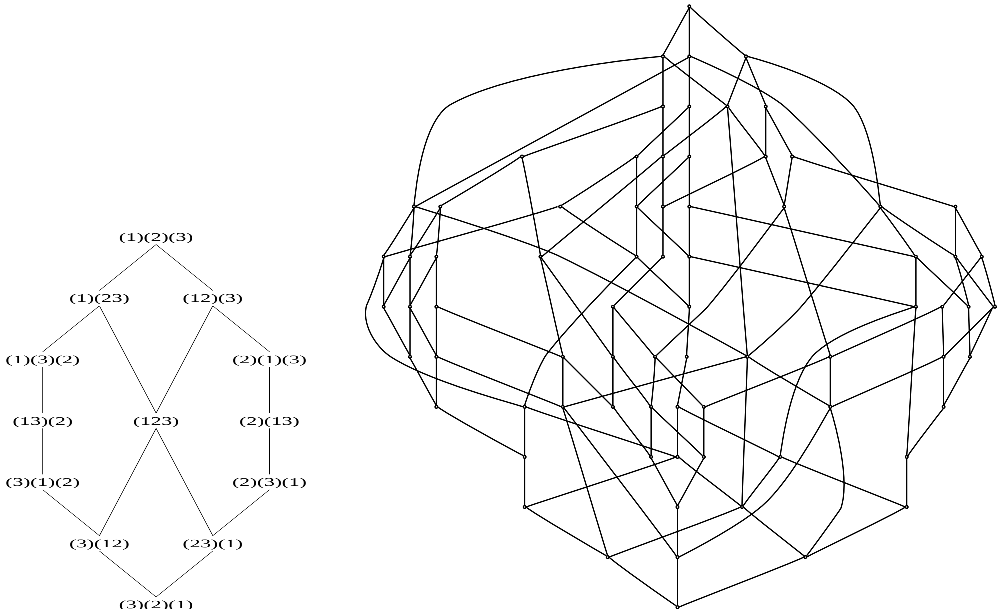


FIGURE 2. The graphs $\mathfrak{G}(3)$ (left) and $\mathfrak{G}(4)$ (right). The orientation of the edges is not displayed since it is always from the topmost vertex to the other.

## 4. Combinatorial properties of $\mathfrak{P}(n)$

In this section, we give enumeration results, combinatorial properties and structural properties of the set $\mathfrak{P}(n)$. Let us begin with the most simple enumerations. We remind to the reader that we will not give the full proofs of the properties since they are long, technical, and do not give any further insight in the structure of pseudo-permutations. They will be published in a longer version of this paper. However, we will always give a few hints about the proofs, so that the reader can believe in the claims.

4.1. **Enumeration of $\mathfrak{P}(n)$.** As defined, the set $\mathfrak{P}(n)$ is the set of ordered partitions of $n$ elements with nonempty parts. Therefore, it satisfies the induction relation:

$$\mathrm{Card}(\mathfrak{P}(\mathrm{n})) = \sum_{\mathrm{i}=0}^{\mathrm{n}-1} \binom{\mathrm{n}}{\mathrm{i}} \mathrm{Card}(\mathfrak{P}(\mathrm{i})).$$

One can also transform the set $\mathfrak{P}(n)$ in order to find a direct (well-known) formula, related to the Eulerian numbers.

Let $n$ be an integer. As already noticed, the order of the elements in a given parenthesis is irrelevant, so we can assume that they are written in increasing order.

**Definition 1.** *Given an element $\mathfrak{s}$ of $\mathfrak{P}(n)$, one can then define its* horizontal reading*, denoted as $\mu(\mathfrak{s})$, which is the permutation obtained by removing all the parentheses, reading the integers in the parentheses in increasing order.*

For example, the horizontal reading of (1)(34)(2)(5) is 13425. One can then consider the classes of the pseudo-permutations which have the same horizontal reading: for example, the pseudo-permutations (123), (1)(23), (12)(3), and (1)(2)(3) form a class of $\mathfrak{P}(n)$. It is clear that, given a permutation $\sigma$, the number of pseudo-permutations in the corresponding class is equal to the number of correct sets of parentheses one can generate. These sets correspond to compositions of $n$ and one can easily see that a composition $I = (i_1, \ldots, i_p)$ is "correct" iff the descent set of $\sigma$ is included in the set $\{i_1, i_1+i_2, \ldots, i_1+\cdots, i_p\}$. It then comes that the number of correct parentheses for a given permutation $\sigma$ of order $n$ is equal to $2^{n-k-1}$ where $k$ is the number of descents of $\sigma$.

Finally, since the number of pseudo-permutations generated from a permutation only depends on the order of the descent set of this permutation, one makes the connection with the Eulerian numbers and one derives the following formula for the cardinality of $\mathfrak{P}(n)$:

$$\mathrm{Card}(\mathfrak{P}(\mathrm{n})) = \sum_{\mathrm{k}=0}^{\mathrm{n}-1} \mathrm{A}_{\mathrm{n,k}}\, 2^{\mathrm{n-k-1}} = \sum_{\mathrm{k}=0}^{\mathrm{n}-1} \mathrm{A}_{\mathrm{n,k}}\, 2^{\mathrm{k}}.$$

4.2. **Inversions.** In this subsection, we generalize to the pseudo-permutations the well-known statistics of inversions over usual permutations.

4.2.1. *Table of inversions.* Let $n$ be an integer and let $\mathfrak{s}$ be an element of $\mathfrak{P}(n)$. For every pair $(i,j)$ with $1 \le i < j \le n$, we define the value of the inversion $(i,j)$ as follows:

- If $i$ and $j$ are in the same parenthese, this value is equal to $\frac{1}{2}$.
- If $i$ and $j$ are in distinct parentheses, this value is equal to $0$ if the parenthese of $i$ is before the parenthese of $j$ in $\mathfrak{s}$ and it is equal to $1$ in the other case.

**Note 2.** *In order to simplify the presentation of the next properties, we define the inversions of $(j,i)$ with $j > i$ as the inversions of $(i,j)$, even if these types of "formal" inversions will not be considered in the table of inversions of the pseudo-permutations.*

The *table of inversions* of $\mathfrak{s}$ is then the list of the non zero-valued pairs $(i,j)$ with $1 \le i < j \le n$ of integers given with the value of the inversion $(i,j)$. The *number of inversions* of $\mathfrak{s}$ is then the sum for all $1 \le i < j \le n$ of the inversions $(i,j)$. For example, the table of inversions of the pseudo-permutation (45)(13)(2) is equal to:

$$\left\{\frac{1}{2}(1,3),\ (1,4),\ (1,5),\ (2,3),\ (2,4),\ (2,5),\ (3,4),\ (3,5),\ \frac{1}{2}(4,5)\right\},$$

and its number of inversions is 8.

**Definition 3.** *Let $\mathfrak{s}$ and $\mathfrak{t}$ be two pseudo-permutations. One says that the table of inversions of $\mathfrak{s}$ is included in the table of $\mathfrak{t}$ if one has:*

$$\forall 1 \le i < j \le n,\ (i,j)_{\mathfrak{s}} \le (i,j)_{\mathfrak{t}}\,.$$

4.2.2. *Characterization of the inversion sets.* One can now ask about the reverse problem: given an inversion table, does it exist a pseudo-permutation that has this inversion table and when the answer is yes, how can one rebuild the corresponding pseudo-permutation.

As for the usual permutations, the answer to the first question is negative in general and one can get the following characterization which generalizes the characterization of the correct sets for the usual inversion tables:

**Proposition 4.** *Let $n$ be an integer and $L$ be a list of the pairs $(i,j)$, with $1 \le i < j \le n$, each pair having a value in the set $\{0, \frac{1}{2}, 1\}$. Then $L$ is the inversion table of a pseudo-permutation if and only if each of the following implications holds:*

- *If the value of an element $(i,j)$ of $L$ is $\frac{1}{2}$, then*
  - *If $k \in [\![\, i,j \,]\!]$, the sum of the value of $(i,k)$ and the value of $(k,j)$ is equal to $1$ in $L$.*
  - *If $k$ is not in $[\![\, i,j \,]\!]$, the value of $(i,k)$ is equal to the value of $(k,j)$ in $L$.*
- *If the value of two elements $(i,j)$ and $(j,k)$ of $L$ are both equal to $1$, and if $i < j < k$ then the value of $(i,k)$ is also $1$ in $L$.*
- *If the value of an element $(i,k)$ of $L$ is $1$, then for every $j \in [\![\, i,k \,]\!]$, either the value of $(i,j)$ or the value of $(j,k)$ is also $1$ in $L$.*

*Proof —* We will not give the complete proof of this theorem, but let us see why all these conditions are necessary.

- The first condition studies all the cases when $i$ and $j$ are in the same parenthesis.
- The second condition says that if $j$ precedes $i$ and $k$ precedes $j$, then $k$ precedes $i$.
- The third condition says that if $k$ precedes $i$ then, whatever the position of $j$ is, either $k$ precedes $j$ or $j$ precedes $i$.

The end of the proof rebuilds from a set that satisfies the three conditions the corresponding pseudo-permutation. This step is very technical, so it is omitted here. ■

**Theorem 5.** *Let $\mathfrak{s}$ and $\mathfrak{t}$ be two pseudo-permutations. Then $\mathfrak{s} < \mathfrak{t}$ if and only if the table of inversions of $\mathfrak{t}$ is included in the table of inversions of $\mathfrak{s}$.*

*Proof —* The proof of this theorem is very technical and we will omit it since we will only need its statement in the sequel. ■

4.3. **Rises of elements of $\mathfrak{P}(n)$.** Let $\mathfrak{s}$ be a pseudo-permutation. We define its *rise set* and its *rise number* respectively as the rise set and the rise number of its horizontal reading. One then has the following property.

**Proposition 6.** *Let $\mathfrak{s}$ be a pseudo-permutation of order $n$. Then the number of outgoing edges of $\mathfrak{s}$ in the pseudo-permutohedron $\mathfrak{G}(n)$ is equal to the rise number of $\mathfrak{s}$.*

*Proof —* This proposition can be proved, first showing that the maximal increasing sequences play disjoint roles, so that one can concentrate on a given increasing sequence and make the proof by induction. ■

**Remark 7.** *One can also provide a dual property that gives the number of incoming edges of a pseudo-permutations in terms of its descent number. But one has to define the descent number in a sligthly different way (that is, write the numbers in the parentheses in* decreasing *order and take the descent number of the horizontal reading).*

With the help of one of the previous properties, one can easily establish the following enumeration result:

**Corollary 8.** *Let $n$ be an integer. The number of edges of the pseudo-permutohedron of order $n$ is given by:*

$$\sum_{k=0}^{n-1} A_{n,k}\, 2^k\, k.$$

## 5. Lattice properties of $\mathfrak{P}(n)$

5.1. **$\mathfrak{G}(n)$ is a lattice.** In this subsection, we prove that the pseudo-permutohedron is a lattice. To prove this property, we first define the infimum and the supremum of two given elements. We actually use the fact that the graph of the order $\mathfrak{G}(n)$ is nothing but the inversion graph of $\mathfrak{P}(n)$ (see Theorem 5) and we prove that this order is a lattice.

To achieve this, we introduce the notion of the trans-union of two given pseudo-permutations. First consider the table of inversions of these elements; make then the union of these tables, taking the maximum value for each pair $(i,j)$; finally compute the transitive closure of this table, that is the smallest table that contains the union and satisfies the first two properties of Theorem 4. This last table is the *trans-union* of the two elements. For example, if we consider (1)(23) and (12)(3) with inversion tables respectively $\{\frac{1}{2}(2,3)\}$ and $\{\frac{1}{2}(1,2)\}$, their trans-union is $\{\frac{1}{2}(1,2), \frac{1}{2}(2,3), \frac{1}{2}(1,3)\}$. Then one can state the following result:

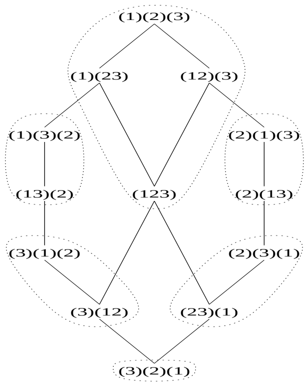


FIGURE 3. $\mathfrak{G}(3)$ is a disjoint union of hypercubes.

**Proposition 9.** *Let us consider two elements of a pseudo-permutohedron. Then their trans-union table is the inversion table of a pseudo-permutation that belong to the same pseudo-permutohedron. Moreover, this element is their infimum.*

*Proof —* This proof has two parts that are not very complicated, but quite long and technical. First, one checks that the trans-union satisfies the three properties of Theorem 4, so that it is the inversion table of a pseudo-permutation. Then one proves that this table is included in the table of every element that is smaller than both elements, and finally conclude that it is the greatest such element (by theorem 5), *i.e.*, the infimum. ■

As it is the case for other lattices, one cannot define the supremum of two elements in such a simple way. However, the pseudo-permutohedron has an interesting property that we can use for this purpose. If one says that the dual of an element is obtained by reading it from right to left, then the graph obtained from $\mathfrak{G}(n)$ by replacing each pseudo-permutation by its dual and by reversing each edge is $\mathfrak{G}(n)$ itself. In other words, $\mathfrak{G}(n)$ is *auto-dual*. Then one obviously has the following duality result:

**Proposition 10.** *Let us consider two elements of a pseudo-permutohedron. Their* supremum *is the dual of the infimum of their duals.*

We can therefore state the main result of this section:

**Theorem 11.** *For all integer $n$, the pseudo-permutohedron of order $n$ is a lattice.*

5.2. **Cutting the pseudo-permutohedron.** Let us consider a pseudo-permutohedron and let us consider the connected components obtained by removing the edges which correspond to the $S$ operators. Then each connected component is an hypercube that contains all the elements of the pseudo-permutohedron which have the same horizontal reading. The bottom-most element is built from this permutation (the horizontal reading) by cutting it as the concatenation of the smallest possible number of increasing sequences and putting these sequences into the same parentheses. The top-most element is also built from this permutation by putting each integer in a different parenthesis. For example, if the horizontal reading is 1673425, then the bottom-most element is (167)(34)(25) and the top-most element is (1)(6)(7)(3)(4)(2)(5).

The dimension of the hypercube is equal to the number of descents of its horizontal reading, *i.e.*, the number of ingoing edges of its elements. As an example, we give the decomposition of $\mathfrak{P}(3)$ as an union of hypercubes in Figure 3.

As we saw before, the pseudo-permutohedron is auto-dual, therefore all the properties one can establish for it can be dualized. Since the $M$ edges and the $S$ edges are exchanged in this process, one can derive that the number of $M$ edges is the half of the total number of edges of $\mathfrak{G}(n)$. The previous results then allows to make in another way the enumeration of the edges of $\mathfrak{G}(n)$.

## Conclusion

In this paper, we have shown a few properties of a new combinatorial object. These properties are fundamental since they are generalizations of the key properties of the symmetric group, which allows us to think that many other statistics defined on the symmetric group have a real meaning in this context. We believe that the pseudo-permutohedron is an interesting and meaningful extension of the permutohedron, and we hope that some work will be done to extend the classical results to this case. It would certainly be interesting to find a geometrical interpretation of the pseudo-permutohedron.

Another interesting direction of investigation with many applications is to consider pseudo-permutations with multiple occurences of the integers. This gives a very general and powerful model which makes it possible to represent the relations between multiple events which can have a temporal length. It seems that most of the results presented here are still valid in this context and some new questions arise.

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