ON THE NUMBER OF CONNECTED COMPONENTS IN THE SPACE OF M-POLYNOMIALS IN HYPERBOLIC FUNCTIONS

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Abstract. We calculate the number of connected components in the space of the so-called M-polynomials of a given degree in hyperbolic functions. By definition an M-polynomial is characterized by the condition that all its critical values are real and distinct.

Résumé. On calcule le nombre de composantes connexes de l'espace des M-polynômes en fonctions hyperboliques d'un degré donné. Par définition, un M-polynôme est caractérisé par la condition que toutes ses valeurs critiques soient réelles et distinctes.

1. Introduction

The present paper continues the series [1–5] dealing with the topology of the sets of usual and trigonometric polynomials with the maximal possible number of real critical values. Following [1–3], we call a (trigonometric) polynomial with he maximal possible number of real critical values a (*trigonometric*) *M-polynomial.* We say that an M-polynomial is *generic* if all its finite critical values are distinct. We call two rational functions (in particular, two polynomials) f_1 and f_2 *equivalent* if $f_1 = f_2 \circ L$, where L is an arbitrary linear fractional transformation of the independent variable. The following results serve as prototypes for the main theorem of this note.

Theorem A (see [2]). *The number $\sharp_n^{pol}$ of connected components of the set of all generic degree n M-polynomials coincides with the number of nonequivalent real degree n polynomials with the same set of $n-1$ real distinct critical values. The latter number equals the number K_{n-1} of up-down permutations of length $n-1$. These numbers are called Euler-Bernoulli numbers and their generating function is*

$$\sum_{n=0}^{\infty} K_n \frac{t^n}{n!} = \sec t + \tan t.$$

Theorem B (see [4, 5]). *The number $\sharp_n^{trig}$ of connected components in the set of trigonometric generic degree n M-polynomials coincides with the number of nonequivalent real trigonometric degree n polynomials with the same set of $2n$ distinct real critical values and equals the Euler number $E_n = K_{2n}$ of up-down permutation of length $2n$. Its exponential generating function is given by*

$$\sum_{n=0}^{\infty} E_n \frac{t^{2n}}{2n!} = \sec t.$$

In the present paper we study a similar question for the case of polynomials in $\sinh x$ and $\cosh x$. Let Hyp_n be the linear space of real degree at most n polynomials in $\sinh x$ and $\cosh x$, i.e. the space of degree at most n polynomials in variables u, v where $u = \sinh x$ and $v = \cosh x$. We call such polynomials *exponential* (in a number of papers in approximation theory they are called hyperbolic, but we will not use this term here to avoid terminological confusion). Note that $\dim \mathrm{Hyp}_n$ equals $2n$, while the dimension of the space of degree at most n polynomials in two variables equals $\binom{n+2}{2}$. One can check that any exponential degree n polynomial has at most $2n-1$ real critical values. Let $\widetilde{D}_n$ denote the set of exponential polynomials $h \in \mathrm{Hyp}_n$ with multiple critical values. Denote by D_n the restriction of the analytic continuation of $\widetilde{D}_n$ in the real space Hyp_n. *Generic exponential*

$$\begin{array}{ccccccc} & & & e_{11} & & & \\ & & e_{22} & & e_{21} & & \\ & e_{31} & & e_{32} & & e_{33} & \\ e_{44} & & e_{43} & & e_{42} & & e_{41} \end{array} \qquad \begin{array}{ccccccc} & & & a_{00} & & & \\ & & a_{10} & & a_{01} & & \\ & a_{20} & & a_{11} & & a_{02} & \\ a_{30} & & a_{21} & & a_{12} & & a_{03} \end{array}$$

a) b)

FIGURE 1. Fig. 1. Euler–Bernoulli triangle after Entringer and Arnold

polynomials are the polynomials in Hyp_n that do not belong to D_n, i.e. their all critical values are real and distinct. Denote the set of all generic exponential M-polynomials of degree n by MHyp_n. Note that for generic exponential M-polynomials, besides the obvious condition that all their critical values are distinct, one has an additional requirement given explicitly below before Lemma 1.

Now we can formulate the main result of the present note.

Theorem 1. *The number $\sharp_n^{hyp}$ of connected components of the set of all generic exponential M-polynomials equals*

$$\sharp_n^{hyp} = \sum_{k=1}^{n}\sum_{r=1}^{n}(k+r-1)\binom{k+r-2}{k-2}\binom{2n-k-r}{n-k}e_{nk}e_{nr},$$

where e_{pq} is the number of up-down permutations of length p starting with the number q.

Remark. The numbers e_{pq} are called *Entringer numbers* and were introduced in [6]. Recall that they form the standard Euler–Bernoulli triangle described in details in [1, 2]. In this triangle the numbers e_{pq} are located in the qth row from left to right in case of odd p, and from right to left in the case of even p (see [7] and Fig. 1a), thus $\sum_{q=1}^{p} e_{pq} = K_p$. In [1, 2] one uses another parametrization of the Euler–Bernoulli triangle, see Fig. 1b. The exponential generating function for the sequence $a_{ij}(-1)^{(i+j)(i+j-1)/2}$ equals $e^y/\cosh(x+y)$. Using these parameters the expression for $\sharp_n^{hyp}$ can be rewritten as

$$\sharp_n^{hyp} = \sum_{i+j=n-1}\ \sum_{m+l=n-1}\left(n+(-1)^n(i-l)\right)\binom{i+m}{i}\binom{j+l}{j}a_{ij}a_{ml}.$$

Note that an exponential polynomial belonging to Hyp_n can be written as $\sum_{i=-n}^{n} a_i e^{ix}$. According to the philosophy formulated in [8], the natural generalization of such polynomials are the *exponential bipolynomials* $\sum_{i=-p}^{q} a_i e^{ix}$, where $p+q=2n$, $p,q>0$, and all a_i are real. The set of all such bipolynomials is denoted by $\widehat{\mathrm{Hyp}}_n$. In analogy with the previous consideration one can define the discriminant $\widehat{D}_n \subset \widehat{\mathrm{Hyp}}_n$, as well as the set of generic exponential M-bipolynomials $\widehat{\mathrm{MHyp}}_n$. The analog of Theorem 1 is as follows.

Theorem 2. *The number $\widehat{\sharp}_n^{hyp}$ of connected components of the set of exponential generic M-bipolynomials equals*

$$\widehat{\sharp}_n^{hyp} = \frac{4n^2-9n+6}{2}K_{2n-1} - 2\sum_{T_n}\sum_{j=1}^{n-2} i_j \prod_{j=1}^{n-2}(i_j-2j+2)(i_j-2j+1),$$

where T_n is the set of all sequences $i_1,\dots,i_{n-2}$, satisfying the inequalities $1 \leqslant i_1 < i_2 < \cdots < i_{n-2} \leqslant 2n-3$.

Remarks. 1. Recall that the number K_{2n-1} is always even (see e.g. [3]).

2. By [9], $2\sum_{T_n}\prod_{j=1}^{n-2}(i_j-2j+2)(i_j-2j+1) = K_{2n-1}$ (see also Lemma 3 below).

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2. General constructions

In order to prove the main result we need the following general construction. Let $f:\ \mathbb{C}P^1 \to \mathbb{C}P^1$ be a real rational function with all real critical values (an *M-function*). Consider the set $C(f)$ of its (geometrically distinct) critical values. We associate to each critical value c its Young diagram λ_c formed by the multiplicities of all critical point whose critical value is c. One can easily check that $\sum_{c\in C(f)} |\lambda_c| = 2n-2$, where n is the degree of f and $|\lambda|$ is the area of the Young diagram λ. An unordered k-tuple $(\lambda_c)_{c\in C(f)}$ will be called the *type* of the function f and denoted by $t(f)$.

Following [10], we define the *net* $S(f)$ of the function f as the inverse image of the real axis $\mathbb{R}P^1 \subset \mathbb{C}P^1$. Evidently, $S(f)$ contains $\mathbb{R}P^1$ and is invariant wrt the complex conjugation. Moreover, all the critical points of f are real as well; they belong to $\mathbb{R}P^1 \subset S(f)$ and are the branching points of $S(f)$ over $\mathbb{R}P^1$. Each critical point is adjacent to an even positive number of arcs (edges) belonging to $S(f)$. These arcs do not intersect outside $\mathbb{R}P^1 \subset S(f)$ and split into pairs invariant under complex conjugation.

A *generalized plane chord diagram* (in what follows just a *chord diagram*) is a circle embedded in $\mathbb{R}^2$ with a fixed finite number of points (*vertices*) some of which are connected by nonintersecting chords and each vertex is adjacent to at least one of the chords. Each chordless cycle of a (generalized plane) chord diagram is called a *face*. *A coloring* of a chord diagram X is a decomposition of the set of its vertices into nonempty subsets (*colors*) such that any pair of vertices lying in the same face is colored into different colors. The set of colors is denoted by $C(X)$. To each color there corresponds a Young diagram μ_c formed by the numbers of chords adjacent to the vertices in this subset. The tuple $(\mu_c)_{c\in C(X)}$ will be called the *type* of the colored chord diagram and denoted by $t(X)$.

A chord diagram is called *oriented* if for every its face one fixes the orientation of its boundary such that each chord is traversed in the same direction on both adjacent faces sharing the considered edge. Thus each chord diagram admits exactly two orientations.

Let X be an oriented colored chord diagram. Traversing each face Γ in the direction prescribed by its orientation in X, we get the induced cyclic order O_Γ on the corresponding subset of the set of colors $C_\Gamma \subseteq C(X)$.

Let us associate to a net $S(f)$ the colored generalized plane chord diagram $X(f)$ defined as follows. The number of vertices in $X(f)$ equals the number of critical points of f; each vertex is labeled by the corresponding critical point and the cyclic order on the set of labels (while traversing the cycle counterclockwise) coincides with the cyclic order of the critical points when we traverse the real line $\mathbb{R}P^1 \subset S(f)$ in the image in the positive direction, i.e. counterclockwise. Two vertices are connected by a chord if and only if the corresponding critical points are connected by an arc belonging to $S(f)$. We assign the same color to two vertices of the diagram $X(f)$ if the critical points labeling these vertices correspond to the same critical value.

Realization Theorem. *Let X be an oriented colored generalized plane chord diagram, C be a finite subset of $\mathbb{R}P^1$ with a cyclic order O induced by the positive orientation of $\mathbb{R}P^1$, $t=(\lambda_c)_{c\in C}$ be the type of real rational degree n M-function. A function f of type t with $C(f)=C$ and $X(f)=X$ exists if and only if there exists a bijection $\varphi:\ C(X)\to C$ satisfying the following conditions:*

1) $\mu_c = \lambda_{\varphi(c)}$ for each color $c\in C(X)$;

2) for each face Γ, the cyclic order on $\varphi(C_\Gamma)$ induced by O_Γ coincides with the restriction of O to $\varphi(C_\Gamma)$.

We say that X *realizes* type t if X satisfies the conditions 1) and 2) for the pair (C,t).

The proof of the Realization Theorem is completely parallel to the proofs of similar results in [10, 11]. We provide a sketch below. The connected components in $\mathbb{C}P^1 \setminus S(f)$ are called the *faces* of the net $S(f)$. Obviously, the faces of the net are splitted into complex conjugate pairs, and each such pair corresponds to a single face in $X(f)$. Using a result by Caratheodory on the correspondence of the boundaries under conformal map, one can easily check that

a) f maps homeomorphically each face to one of the hemispheres in $\mathbb{C}P^1 \setminus \mathbb{R}P^1$;

b) f maps homeomorphically the boundary of each face of the net to $\mathbb{R}P^1$ (see e.g., [10]).

Conditions a) and b) imply that the colored chord diagram corresponding to $S(f)$ satisfies 1) and 2) in Realization Theorem. In order to verify that for any oriented colored chord diagram X satisfying 1) and 2) there exists a rational function with the equivalent net, let us construct, using X, a branched covering satisfying a) and b). Having done that we induce the complex structure on the preimage from that on the image, and the topological branched covering is transformed into a holomorphic one. The construction of the topological covering goes as follows. Note that it suffices to define it on the disc containing the chord diagram, and then to glue the map $\mathbb{C}P^1$ to $\mathbb{C}P^1$ by pasting together the initial map and its complex conjugate along the real axis. In order to determine our map on the disc, we first map the vertices of the chord diagram to the corresponding critical values (using the bijection φ). Then we map the chords connecting the vertices to the arcs of $\mathbb{R}P^1$ between the corresponding critical values, taking into account their orientation. Finally, the faces of the chord diagram are mapped homeomorphically to the upper or lower hemispheres according to their orientations. $\square$

We say that two oriented colored chord diagrams are *equivalent* if there exists a homeomorphism (possibly inverting the orientation) of the planes containing the diagrams that identifies the diagrams themselves, i.e. sends cycles to cycles, chords to chords, and vertices of different colors to vertices of different colors. Let us now fix the set C and the type $t = (\lambda_c)_{c \in C}$.

Corollary 3. *The number of nonequivalent real rational functions with the set of critical values C and type t equals the number of nonequivalent oriented colored generalized plane chord diagrams realizing the type t.*

3. Exponential M-polynomials

Let us now pass to the case of real polynomials in $\sinh x$ and $\cosh x$. Note that $\cosh^2 x - \sinh^2 x = 1$, and while x traverses the whole real axis, the curve $(\cosh x, \sinh x)$ traverses the right branch of the hyperbola $v^2 - u^2 = 1$. Thus, any polynomial p_n of degree n in $\sinh x$ and $\cosh x$ is obtained by the restriction of the corresponding polynomial P_n of degree n in variables (u, v) to the right branch of the hyperbola. Consider the restriction of P_n to the whole hyperbola $v^2 - u^2 = 1$, and change the variables to $y = u + v = e^x$, $z = v - u = e^{-x}$. In these new variables the hyperbola is given by $yz = 1$ and the restriction of P_n to this curve after elimination of z coincides with $\tilde{p}_{2n}(y)/y^n$, where $\tilde{p}_{2n}$ is some polynomial in y of degree $2n$ with real coefficients. Taking the derivative one gets that the total number of the critical points of the expression $\tilde{p}_{2n}(y)/y^n$ different from 0 and ∞ (and thus coinciding with the number of its finite critical points) equals $2n$.

Denote by Δ_n the connected set of all unordered n-tuples of pairwise distinct real numbers. Let us define the *generalized Lyashko-Looijenga mapping*

$$LL: \ \mathrm{MHyp}_n \to \Delta_{2n}$$

sending each exponential generic M-polynomial p to the unordered set of the critical values of its analytic continuation to the whole hyperbola. (Its restriction to the other branch of hyperbola one can easily obtain by replacing $\sinh x$ in the polynomial p by $-\sinh x$; our generity condition mentioned in the introduction is that the real critical values of the obtained polynomial are distinct and different from the critical values of the original polynomial as well.) Note that LL is the mapping between the spaces of the same dimension, and as follows directly, from the above corollary, the number of inverse images of any point in Δ_{2n} is a constant. Let us denote it by $\flat_n$.

Lemma 1. *The map $LL:\ \mathrm{MHyp}_n \to \Delta_{2n}$ is a real covering of degree $\sharp_n^{hyp}$. Therefore, the number $\sharp_n^{hyp}$ of the connected components in the space MHyp_n equals the number $\flat_n$ of points in the fiber.*

Proof. Denote by $\mathcal{P}_{2n}$ the set of all rational functions of the form $p_{2n}(y)/y^n$ with real coefficients. According to the above condition, there exists a well-defined bijection $\Psi_n:\ \mathrm{Hyp}_n \to \mathcal{P}_{2n}$. The image MHyp_n under this bijection we denote by $M\mathcal{P}_{2n} \subset \mathcal{P}_{2n}$. Obviously it suffices to prove our statement for the map

$$\widetilde{LL}:\ M\mathcal{P}_{2n} \to \Delta_{2n}.$$

Let us show that

a) the map $\widetilde{LL}$ is open, i.e. it is a local homeomorphism onto the image;

b) the map $\widetilde{LL}$ is surjective.

Properties a), b) show that $\widetilde{LL}:\ M\mathcal{P}_{2n} \to \Delta_{2n}$ is a covering, and the number of points in the inverse image of any point from Δ_{2n} coincides by definition with $\flat_n$. The property a) is fairly obvious. Indeed, choose an arbitrary function $f = p_{2n}(y)/y^n \in M\mathcal{P}_{2n}$. Let $\{y_1,\dots,y_{2n}\}$ be the set of its critical points distinct from 0 and ∞. Since derivation is linear, one can find for any i between 1 and $2n$ the polynomial $q_{2n,i}(y)$ such that the rational function $q_{2n,i}(y)/y^n$ has all y_j, $j=1,\dots,2n$, $j\neq i$, as its critical points.

Consider the 1-parameter family $\Pi_i(\gamma) = (p_{2n}(y)+\gamma q_{2n,i}(y))/y^n$, where the absolute value of γ is small, and calculate the tangent vector to the curve $\widetilde{LL}(\Pi_i(\gamma))$ at $\gamma=0$. Let $\{y_1(\gamma),\dots,y_{2n}(\gamma)\}$ and $\{c_1(\gamma),\dots,c_{2n}(\gamma)\}$ be the set of its critical points and the corresponding critical values in the family $\Pi_i(\gamma)$. Obviously, $dy_j(0)/d\gamma = 0$ for $j\neq i$, and $dy_i(0)/d\gamma = \kappa \neq 0$. This implies immediately that $dc_j(0)/d\gamma = 0$ for $j \neq i$, but $dc_i(0)/d\gamma = \kappa dq_{2n,i}(y_i)/dy \neq 0$. Thus only the ith critical value deforms in the family while the rest of the critical values are preserved. Property b) follows immediately from the above Realization Theorem. □

Thus, our problem is reduced to the calculation of the number $\flat_n$. Note that the type of any generic function in $\mathcal{P}_{2n}$ contains exactly one Young diagram of the form $(n-1,n-1)$, as well as $2n$ diagrams of the form (1). The above corollary implies that such functions correspond to the diagrams with $2n+2$ vertices containing a pair of equally colored vertices v_0, v_0' adjacent to exactly $n-1$ chords, while the other $2n$ vertices $v_1,\dots,v_{2n}$ are colored into pairwise distinct colors different from that used for v_0, v_0', and moreover, each of them is incident to just one chord. The pair v_0, v_0' cannot be joined by a chord, since vertices of the same color cannot belong to the same face. Therefore, there exists just one pair of vertices v_i, v_j, $i,j>0$, connected by a chord, while all the remaining vertices v_k, $k>0$ are connected either to v_0 or to v_0'. Moreover, the vertex pairs (v_i,v_j) and (v_0,v_0') separate each other on the cycle of the chord diagram, since otherwise the equally colored vertices v_0 and v_0' would belong to the same face, which is impossible. Finally, two arcs of the cycle with the endpoints v_0 and v_0' correspond to two branches of the hyperbola. Thus, the chord diagrams corresponding to the functions from $M\mathcal{P}_{2n}$ enjoy the following additional requirement: all the vertices v_k, $k\neq 0,i,j$, lie on the same arc between v_0 and v_0'. Without loss of generality we can assume that the cyclic ordering of vertices while traversing the circle clockwise is as follows: $v_0,v_1,v_2,\dots,v_{2n-1},v_0',v_{2n}$, and therefore v_i and v_j are v_n and v_{2n} respectively (see Fig. 2).

Let us label all the finite critical values of a function belonging to $M\mathcal{P}_{2n}$ using numbers 1 to $2n$ in the increasing order, and let us label the critical value ∞ by $2n+1$. Then condition 1) of the Realization Theorem is satisfied if one considers instead of the bijection φ the function ψ on the set of vertices of the chord diagram with values in $\{1,\dots,2n+1\}$ that maps bijectively the vertices $v_1,\dots,v_{2n}$ to $\{1,\dots,2n\}$ and takes the value $2n+1$ at v_0 and v_0'. Fix one of the two possible orientations of the chord diagram and assume that with this orientation the arc (v_1,v_2) not containing other vertices is traversed in the direction from v_1 to v_2. Then condition 2) of the Realization Theorem for the face containing this arc is equivalent to the inequality $\psi(v_1)<\psi(v_2)$. For the neighboring face we get the inequality $\psi(v_2)>\psi(v_3)$, etc., the last one being $\psi(v_{2n-2})>\psi(v_{2n-1})$. Besides, considering the faces containing the arc (v_n,v_{2n}), one gets an extra inequality

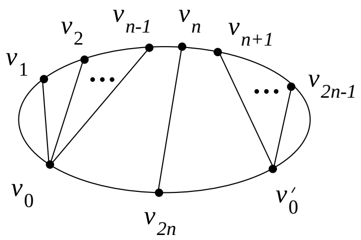

FIGURE 2. Chord diagram of an exponential M-polynomial

$\psi(v_n) < \psi(v_{2n})$ in the case of even n and $\psi(v_n) > \psi(v_{2n})$ in the case of odd n. Reversing the orientation of the chord diagram to the opposite, one should reverse the signs in all the inequalities.

Let us introduce the function $\tilde{\psi}$ on the set of vertices of the oriented chord diagram, defining it by the conditions $\tilde{\psi}(v) = \psi(v)$ if $\psi(v_{2n}) > \psi(v_n)$, and $\tilde{\psi}(v) = 2n + 2 - \psi(v)$ if $\psi(v_{2n}) < \psi(v_n)$. Thus, the numbers $\tilde{\psi}(v_1), \dots, \tilde{\psi}(v_{2n-1})$ form an up-down (if n is even) or a down-up (if n is odd) permutation, and for any n the following conditions are satisfied

$$(*) \quad \tilde{\psi}(v_{2n}) > \tilde{\psi}(v_n), \qquad \tilde{\psi}(v_{2n}) \neq \tilde{\psi}(v_i), \quad i \neq 2n, \qquad \tilde{\psi}(v_n) > \tilde{\psi}(v_{n-1}), \qquad \tilde{\psi}(v_n) > \tilde{\psi}(v_{n+1}).$$

Note that any such permutation occurs twice: once for each orientation of the chord diagram. On the other hand, for a fixed orientation the permutations $(a_1, \dots, a_{2n-1})$ and $(a_{2n-1}, \dots, a_1)$ correspond to equivalent chord diagrams. Therefore, the number $\flat_n$ equals to the number of up-down or down-up permutations satisfying $(*)$. The easiest way to calculate this number is to split the permutation $\pi = (\tilde{\psi}(v_1), \dots, \tilde{\psi}(v_{2n-1}))$ into two down-up permutations (independently on the parity of n) $\pi_r = (\tilde{\psi}(v_n), \tilde{\psi}(v_{n+1}), \dots, \tilde{\psi}(v_{2n-1}))$ and $\pi_l = (\tilde{\psi}(v_n), \tilde{\psi}(v_{n-1}), \dots, \tilde{\psi}(v_1))$ with the common first entry.

Assume that this common entry equals m and that π_r contains exactly k entries smaller than m. Then these k entries can be chosen in $\binom{m-1}{k}$ ways, the element $\tilde{\psi}(v_{2n})$ can be chosen in $2n - m$ ways, and the remaining $n - k - 1$ entries π_r in $\binom{2n-1-m}{n-k-1}$ ways. Thus fixing m and k contributes

$$(2n - m)\binom{m-1}{k}\binom{2n-1-m}{n-k-1} e_{n,k+1} e_{n,m-k}$$

to the total amount of permutations. Adding these summands over m from 1 to $2n - 1$ and over k from 1 to n gives the formula of Theorem 1.

In order to prove Theorem 2, let us introduce, in an analogy with the previous considerations, the Lyashko-Looijenga mapping $\widehat{LL} : \widehat{\mathrm{MHyp}}_n \to \Delta_{2n}$. The following statement is an analog of Lemma 1 and is proved by exactly the same arguments.

Lemma 2. *The map $\widehat{LL} : \widehat{\mathrm{MHyp}}_n \to \Delta_{2n}$ is a real covering of degree $\widehat{\sharp}_n^{hyp}$. Therefore, the number $\widehat{\sharp}_n^{hyp}$ of the connected components in the space $\widehat{\mathrm{MHyp}}_n$ equals the number of points $\widehat{\flat}_n$ in the fiber.*

In order to calculate $\widehat{\flat}_n$, let us study the structure of chord diagrams corresponding to generic functions in the space $\widehat{\mathcal{P}}_{2n}$ of rational functions of the form $p_{2n}(y)/y^p$, $1 \leqslant p \leqslant 2n - 1$. One can easily see that these diagrams differ from those considered above only in the following detail. There are $p - 1$ chords incident to the vertex v_0, and there are $q - 1$ chords incident to v_0'; additionally, in case $p = 1$ ($q = 1$ resp.), the vertex v_0 (the vertex v_0' resp.) is missing. The only chord incident to the vertices v_0 and v_0' connects the vertices v_p and v_{2n}. Defining the function $\tilde{\psi}$ in the same way as above we get that the numbers $\tilde{\psi}(v_1), \dots, \tilde{\psi}(v_{2n})$ satisfy the same conditions with the only distinction that in $(*)$ v_{n-1}, v_n and v_{n+1} are changed by v_{p-1}, v_p and v_{p+1} resp.

To continue our calculations we need the following combinatorial statement. Let $(a_1, \dots, a_{2k+1})$ be a down-up permutation; its entries located at the odd positions are called *maxima*, and those standing at the even positions are called *minima*. One can easily check that the numbers $2k+1$ and $2k$ are always maxima. Let $I = \{i_1 < i_2 < \cdots < i_{k-1}\}$ be the set of all remaining (nontrivial) maxima in the increasing order. Denote by $A^{du}_{k-1}(I)$ the set of all down-up permutations whose set of maxima coincides with I. Similarly, if $(a_1, \dots, a_{2k+1})$ is an up-down permutation , then its maxima are the entries standing at even positions and minima are those at odd positions. The entry $2k+1$ is always a trivial maximum, the remaining maxima are nontrivial. Let us denote by $A^{ud}_{k-1}(I)$ the set of all up-down permutations whose set of nontrivial maxima coincides with $\{i_1+1, i_2+1, \dots, i_{k-1}+1\}$.

Lemma 3. *For all k and I*

$$|A^{du}_{k-1}(I)| = |A^{ud}_{k-1}(I)| = 2\prod_{j=1}^{k-1}(i_j - 2j + 2)(i_j - 2j + 1).$$

Proof. In the case of down-up permutations the statement is proved in [9]. To prove it for the up-down case, let us construct the bijection between the sets $A^{du}_{k-1}(I)$ and $A^{ud}_{k-1}(I)$. Take $(a_1, \dots, a_{2k+1}) \in A^{du}_{k-1}(I)$ with $a_r = 2k+1$. Define $(b_1, \dots, b_{2k+1})$ by conditions

$$b_i = \begin{cases} a_{r+i} + 1, & 1 \leqslant i \leqslant 2k - r + 1, \\ 1, & i = 2k - r + 2, \\ a_{i-2k+r-2} + 1, & 2k - r + 3 \leqslant i \leqslant 2k + 1. \end{cases}$$

One can easily check that $(b_1, \dots, b_{2k+1}) \in A^{du}_{k-1}(I)$. Since the above procedure is invertible the lemma follows. □

Further, one checks that if a permutation $(\tilde{\psi}(v_1), \dots, \tilde{\psi}(v_{2n-1}))$ belongs to $A^{du}_{n-2}(I)$ then its contribution to the total number of permutations equals $3 + \sum_{j=1}^{n-2}(2n - i_j)$ (the number 3 in this formula is the contribution of the trivial maxima $i_{n-1} = 2n-2$ and $i_n = 2n-1$). Similarly, if $(\tilde{\psi}(v_1), \dots, \tilde{\psi}(v_{2n-1}))$ belongs to $A^{ud}_{n-2}(I)$ then its contribution equals $1 + \sum_{j=1}^{n-2}(2n - 1 - i_j)$. Adding these contributions over all possible values of I and taking into account Lemma 3, one gets Theorem 2.

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