PARTY ALGEBRA AND CONSTRUCTION OF ITS IRREDUCIBLE REPRESENTATIONS

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ABSTRACT. Suppose that there exist two parties each of which consists of n members. The parties hold meetings splitting into several small groups. Every group consists of the same number of members of each party. The set of such decompositions into small groups makes an algebra called *the party algebra* under a certain product. We show that the party algebra is semisimple by constructing a complete set of irreducible representations.
RÉSUMÉ. C'est le résumé français.

1. INTRODUCTION

The unitary reflection group $G(r,p,k)$ in the Shephard-Todd notation [6] acts on a vector space V of dimension k by permuting the basis elements $e_1, e_2, \ldots, e_k$. In 1997, Tanabe showed in his paper [7] that the centralizer $\mathrm{End}_{G(r,p,k)}(V^{\otimes n})$ is generated by the symmetric group $\mathfrak{S}_n$—acting on $V^{\otimes n}$ by tensor factors permutation—together with three further special operators.

In 1994, Martin defined the partition algebra $P_n(\beta^2)$ introducing a product among partitions of a set M of size $2n$ [4]. This algebra is generated by the symmetric group $\mathfrak{S}_n$ and two special elements which correspond to two of the above special operators. In fact, in case $\beta \neq 0$ and $k \geq n$, $P_n(\beta^2)$ is isomorphic to the centralizer $\mathrm{End}_{G(1,1,k)}(V^{\otimes n}) = \mathrm{End}_{\mathfrak{S}_k}(V^{\otimes n})$ ($\mathfrak{S}_k$ acts diagonally on $V^{\otimes n}$).

For a positive integer n, the *party algebra* $A_n(\beta)$ is defined as an abstract algebra with a parameter β generated by the symmetric group $\mathfrak{S}_n$ and one of the above operators, which naturally becomes an subalgebra of the partition algebra $P_n(\beta^2)$. A word of the generators of $A_n(\beta)$ reduces to a scalar multiple of a *standard* word of the generators by the relations. Besides, the standard words of $A_n(\beta)$ have one to one correspondences with *the seat-plans* for the meetings held by two parties each of which consists of n members under certain conditions (see Section 2). If $\beta \neq 0$ and $r > n$, then we have a surjective homomorphism from the party algebra onto $\mathrm{End}_{G(r,1,k)}(V^{\otimes n})$. Moreover, if $k \geq n$, then we can show the above homomorphism is injective using Schur-Weyl reciprocity and tensor decomposition. In particular, in case $\beta \neq 0$ we find that the party algebra is semisimple.

In this talk, we explicitly construct a complete set of irreducible representations of the party algebra, extending the orthogonal representations of the symmetric group $\mathfrak{S}_n$.

2. DEFINITION OF THE PARTY ALGEBRAS

We consider the following situation. Let $D = \{d_1, d_2, \ldots, d_n\}$ and $R = \{r_1, r_2, \ldots, r_n\}$ be two sets each of which consists of n distinct elements such that $D \cap R = \emptyset$. We decompose $D \sqcup R$ into subsets $M_1, M_2, \ldots, M_n$ (some of M_js might be empty) so that they satisfy

$$\begin{aligned}
&\bigsqcup_{j=1,2,\ldots,n} M_j = D \sqcup R,\\
&|M_1| \geq |M_2| \geq \cdots \geq |M_n|,\\
&|M_j \cap D| = |M_j \cap R| \text{ for } j = 1, 2, \ldots, n,\\
&M_i \cap M_j = \emptyset \text{ if } i \neq j.
\end{aligned}$$

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We call such a partition into subsets a *seat-plan* of size n. Let $P(n)$ be a set of partitions of n. Then there exists a partition $\lambda \in P(n)$ such that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) = (|M_1|/2, |M_2|/2, \dots, |M_n|/2)$. The number of seat-plans is

$$\sum_{\lambda \in P(n)} \left(\frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_n!} \right)^2 \cdot \frac{1}{\alpha_1! \alpha_2! \cdots \alpha_n!},$$

where $\alpha_i = |\{\lambda_k ; \lambda_k = i\}|$.

A seat-plan of size n is figured as follows. Consider a rectangle with n marked points on the bottom and the same n on the top as in Figure 1. The n marked points on the bottom are labeled

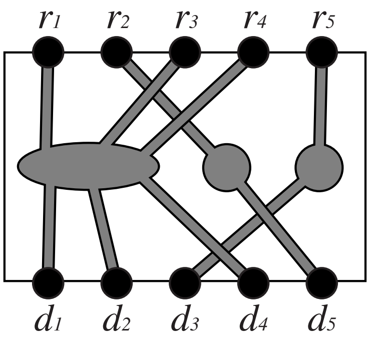

FIGURE 1. A seat-plan

by $d_1, d_2, \dots d_n$ from left to right. Similarly, the n marked points on the top is labeled by $r_1, r_2, \dots, r_n$ from left to right. If $D \sqcup R$ is divided into non-empty m subsets, then put m shaded circles in the middle of the rectangle so that they have no intersections. Each of the circles corresponds to one of the non-empty M_js. Then we join the $2n$ marked points and the m circles with $2n$ shaded bands so that the marked points labeled by the elements of M_j are connected to the corresponding circle with $|M_j|$ bands.

Now we define the product $w_1 w_2$ between two of rectangles w_1, w_2 (each of which corresponds to a seat-plan) by placing w_1 on w_2, gluing the corresponding boundaries and shrinking half along the vertical axis as in Figure 2. We then have a new diagram possibly containing some closed loops. If there are p closed loops, the product is then the resulting diagram, with the closed loops removed, times the scalar β^p. It is easy to define this product in terms of seat-plans. By this definition, a set of linear combinations of seat-plans of size n over $\mathbb{C}$ makes an algebra $A_n(\beta)$. We call it the *party algebra*. The identity of $A_n(\beta)$ is a diagram which corresponds to a partition $D \sqcup R = \{d_1, r_1\} \sqcup \{d_2, r_2\} \sqcup \cdots \sqcup \{d_n, r_n\}$. We put $A_0(\beta) = A_1(\beta) = \mathbb{C}$.

Next, we define special elements $f_1, f_2, \dots, f_{n-1}$ and $s_1, s_2, \dots, s_{n-1}$ of $A_n(\beta)$ as in Figure 3. By the induction on n, we can easily find that those special elements generate $A_n(\beta)$. (See the paper [4].)

We easily have the following relations:

$$\begin{array}{rcll}
s_i^2 &=& \mathrm{id} \quad (i = 1, 2, \dots, n-1), & (P1) \\
s_i s_{i+1} s_i &=& s_{i+1} s_i s_{i+1} \quad (i = 1, 2, \dots, n-2), & (P2) \\
s_i s_j &=& s_j s_i \quad (|i-j| \geq 2, i, j = 1, 2, \dots, n-1), & (P3) \\
f_i^2 &=& \beta f_i \quad (i = 1, 2, \dots, n-1), & (P4) \\
f_i s_i &=& s_i f_i = f_i \quad (i = 1, 2, \dots, n-1), & (P5) \\
f_i f_j &=& f_j f_i \quad (i, j = 1, 2, \dots, n-1), & (P6) \\
s_i f_{i+1} s_i &=& s_{i+1} f_i s_{i+1} \quad (i = 1, 2, \dots, n-2), & (P7) \\
f_i s_j &=& s_j f_i \quad (|i-j| \geq 2, i, j = 1, 2, \dots, n-1). & (P8)
\end{array}$$

The following proposition shows that the above relations define the party algebra.

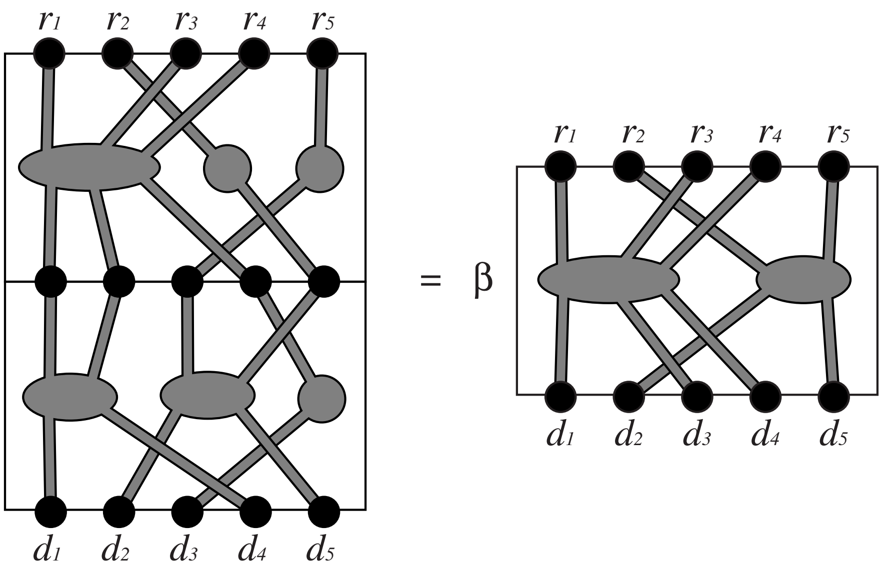

FIGURE 2. The product of seat-plans

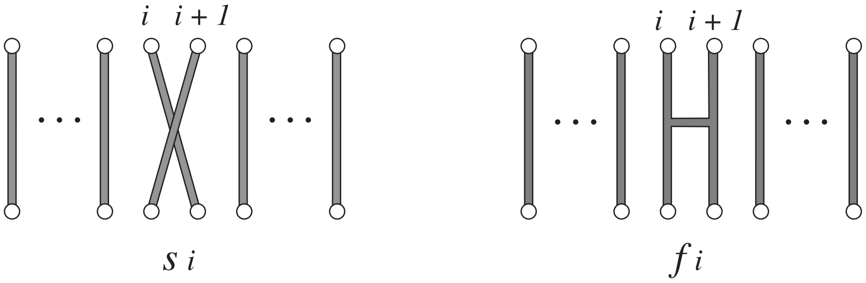

FIGURE 3. Special elements

Proposition 1. *For an integer $n > 1$, the party algebra $A_n(\beta)$ is characterized by the following generators and relations:*

$$
\begin{aligned}
\textit{generators}; \quad & f, s_1, s_2, \ldots, s_{n-1}, \\
\textit{relations}; \quad & s_i^2 = 1 \quad (1 \leq i \leq n-1), \\
& s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (1 \leq i \leq n-2), \\
& s_i s_j = s_j s_i \quad (|i-j| \geq 2), \\
& f^2 = \beta f, \\
& f s_1 = s_1 f = f, \\
& f s_i = s_i f \quad (i \geq 3), \\
& f s_2 f s_2 = s_2 f s_2 f, \\
& f s_2 s_1 s_3 s_2 f s_2 s_1 s_3 s_2 = s_2 s_1 s_3 s_2 f s_2 s_1 s_3 s_2 f.
\end{aligned}
$$

Proof. Put

$$f_1 = f, \quad f_i = (s_{i-1} \cdots s_2 s_1)(s_i \cdots s_3 s_2) f_1 (s_2 s_3 \cdots s_i)(s_1 s_2 \cdots s_{i-1}) \ (i = 2, 3, \ldots, n-1).$$

Then we find the relations in the proposition are equivalent to the relations $(P1)$-$(P8)$. Moreover each word made of f_is and s_is reduces to a scalar multiple of a standard word which denotes a seat-plan. □

The following proposition given by Tanabe [7] shows the relation between the party algebras and the centralizer algebras of the unitary reflection groups.

Proposition 2. *(Tanabe* [7, Theorem 3.1]*) Let $G(r,1,k)$ be the group of all the monomial matrices of size n whose non-zero entries are r-th roots of unity. Let V be a vector space of dimension k with the basis elements $e_1, e_2, \ldots, e_k$ on which $G(r,1,k)$ acts naturally. Let ϕ be the representation of the symmetric group $\mathfrak{S}_n$ on $V^{\otimes n}$ obtained by permuting the tensor product factors, i.e., for $v_1, v_2, \ldots, v_n \in V$ and for $w \in \mathfrak{S}_n$,*

$$\phi(w)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) := v_{w^{-1}(1)} \otimes v_{w^{-1}(2)} \otimes \cdots \otimes v_{w^{-1}(n)}.$$

Define further $\phi(f)$ as follows:

$$\phi(f)(e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_n}) := \begin{cases} k(e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_n}) & \text{if } p_1 = p_2, \\ 0 & \text{otherwise.} \end{cases}$$

If $r > n$, then $End_{G(r,1,k)}(V^{\otimes n})$ is generated by $\phi(\mathfrak{S}_n)$ and $\phi(f)$, and ϕ defines a homomorphism from $A_n(k)$ to $End_{G(r,1,k)}(V^{\otimes n})$.

The above result also holds for $A_n(\beta)$, if we replace k with a non-zero parameter $\beta \in \mathbb{C}$ in the definition of $\phi(f)$. Moreover if $k \geq n$, then we can show that the above ϕ is injective using Schur-Weyl reciprocity and tensor decomposition, and counting the dimension. In particular, for a non-zero parameter $\beta \in \mathbb{C}$ we find that $A_n(\beta)$ is semisimple.

In the following section, we construct a complete set of irreducible representations of $A_n(\beta)$ extending the orthogonal representations of the symmetric group $\mathfrak{S}_n$.

3. Construction

We define representations of $A_n(\beta)$ which turn out to make a complete set of irreducible representations. The representations are constructed on the Bratteli diagram for the sequence $A_0(\beta) \subset A_1(\beta) \subset \cdots \subset A_n(\beta)$ as in Figure 4. (See for example the papers [1, 2, 5, 8, 9].)

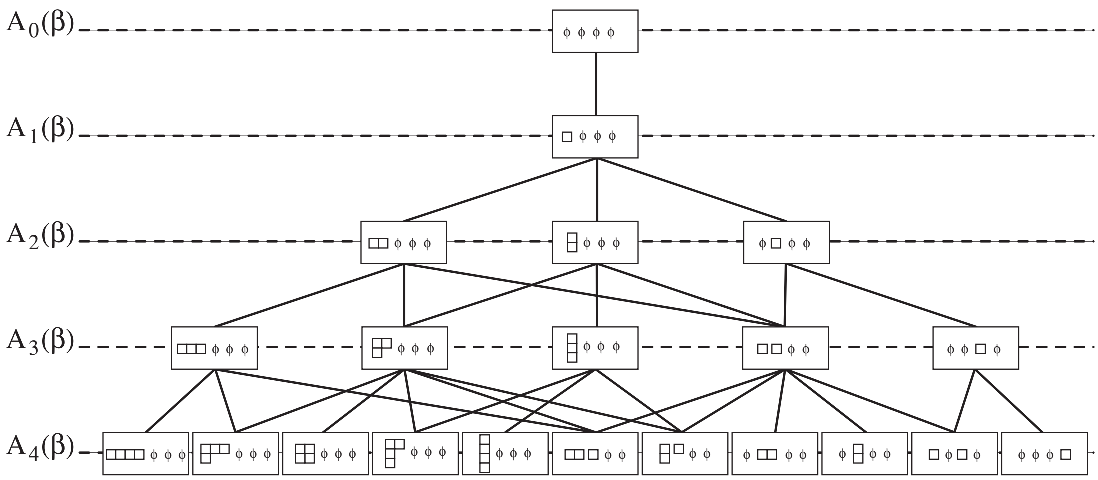

Figure 4. The Bratteli diagram for $A_4(\beta)$

Fix a positive integer n. Let

$$\boldsymbol{\alpha} = [\alpha(1), \ldots, \alpha(n)]$$

be an n-tuple of Young diagrams. The j-th coordinate of the tuple is referred to *the j-th board*. The *height* $\|\boldsymbol{\alpha}\|$ of $\boldsymbol{\alpha}$ is defined as the weight sum of the sizes of all the $|\alpha(j)|$s. Namely, $\|\boldsymbol{\alpha}\|$ is defined by

$$\|\boldsymbol{\alpha}\| = \sum_{j=1}^{n} j|\alpha(j)|.$$

Let

$$\Lambda_n(i) = \{\boldsymbol{\alpha} = [\alpha(1), \ldots, \alpha(n)] \ ; \ \|\boldsymbol{\alpha}\| = i\}$$

be a set of n-tuples of height i. For $\boldsymbol{\alpha} \in \Lambda_n(i)$, we set $\alpha(0) = n - i$ (the horizontal Young diagram of depth 1 and of width $n - i$) if necessary. Let $\boldsymbol{\alpha} \underset{1}{\prec} \tilde{\boldsymbol{\alpha}}$ or $\tilde{\boldsymbol{\alpha}} \underset{1}{\succ} \boldsymbol{\alpha}$ denote that $\tilde{\boldsymbol{\alpha}}$ is obtained from $\boldsymbol{\alpha}$ by removing one box from a Young diagram on the j-th board and adding the box to a Young diagram on the $(j+1)$-st board for some j $(0 \le j \le n-1)$.

For $\boldsymbol{\alpha} \in \Lambda_n(n)$, *tableaux* $\mathbb{T}(\boldsymbol{\alpha})$ *of shape* $\boldsymbol{\alpha}$ is defined by

$$\begin{aligned} \mathbb{T}(\boldsymbol{\alpha}) \quad = \quad & \{P = (\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(n)}); \quad \boldsymbol{\alpha}^{(0)} = [\emptyset, \ldots, \emptyset], \boldsymbol{\alpha}^{(n)} = \boldsymbol{\alpha}, \\ & \boldsymbol{\alpha}^{(i)} \underset{1}{\prec} \boldsymbol{\alpha}^{(i+1)} \text{ for } 0 \le i \le n-1\}. \end{aligned}$$

Let $V(\boldsymbol{\alpha}) = \oplus_{P \in \mathbb{T}(\boldsymbol{\alpha})} \mathbb{C} v_P$ be a vector space over $\mathbb{C}$ with the standard basis $\{v_P | P \in \mathbb{T}(\boldsymbol{\alpha})\}$.

For a generator s_i of $A_n(\beta)$, we define a linear map on $V(\boldsymbol{\alpha})$ giving the matrix A_i with respect to the basis $\{v_P | P \in \mathbb{T}(\boldsymbol{\alpha})\}$. Namely, for a pair of tableaux $P = (\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(n)})$ and $Q = (\boldsymbol{\alpha}'^{(0)}, \boldsymbol{\alpha}'^{(1)}, \ldots, \boldsymbol{\alpha}'^{(n)})$ of $\mathbb{T}(\boldsymbol{\alpha})$ define $A_i v_P = \sum_{Q \in \mathbb{T}(\boldsymbol{\alpha})} (A_i)_{QP} v_Q$. If there is an $i_0 \in \{1, 2, \ldots, n-1\} \setminus \{i\}$ such that $\boldsymbol{\alpha}^{(i_0)} \neq \boldsymbol{\alpha}'^{(i_0)}$, then we put

$$(A_i)_{QP} = 0.$$

In the following, we consider the case that $\boldsymbol{\alpha}^{(i_0)} = \boldsymbol{\alpha}'^{(i_0)}$ for $i_0 \in \{1, 2, \ldots, n-1\} \setminus \{i\}$.

First, we assume that $\boldsymbol{\alpha}^{(i-1)}$ and $\boldsymbol{\alpha}^{(i+1)}$ of a tableau P coincide with each other except on the j-th and the $(j+1)$-st boards. In this case, $\boldsymbol{\alpha}^{(i)}$ is obtained from $\boldsymbol{\alpha}^{(i-1)}$ by moving a box in a Young diagram on the j-th board to a Young diagram on the $(j+1)$-st board and $\boldsymbol{\alpha}^{(i+1)}$ is obtained from $\boldsymbol{\alpha}^{(i)}$ by moving another box in the Young diagram on the j-th board to the Young diagram on the $(j+1)$-st board. Denote the Young diagram on the j-th board of $\boldsymbol{\alpha}^{(i-1)}$ [resp. $\boldsymbol{\alpha}^{(i)}$, $\boldsymbol{\alpha}^{(i+1)}$] by $\lambda^{(i-1)}$ [resp. $\lambda^{(i)}$, $\lambda^{(i+1)}$] and denote the Young diagram on the $(j+1)$-st board of $\boldsymbol{\alpha}^{(i-1)}$ [resp. $\boldsymbol{\alpha}^{(i)}$, $\boldsymbol{\alpha}^{(i+1)}$] by $\mu^{(i-1)}$ [resp. $\mu^{(i)}$, $\mu^{(i+1)}$]. Let $\lambda' \underset{1}{\subset} \lambda$ or $\lambda \underset{1}{\supset} \lambda'$ denote that λ' is obtained from λ by removing one box. Recall that if $\nu \underset{1}{\subset} \mu \underset{1}{\subset} \lambda$, then we can define the *axial distance* $d = d(\nu, \mu, \lambda)$. Namely if μ differs from ν in its r_0-th row and c_0-th column only, and if λ differs from μ in its r_1-th row and c_1-th column only, then $d = d(\nu, \mu, \lambda)$ is defined by

$$d = d(\nu, \mu, \lambda) = (c_1 - r_1) - (c_0 - r_0) = \begin{cases} h_\lambda(r_1, c_0) - 1 & \text{if } r_0 \le r_1, \\ 1 - h_\lambda(r_0, c_1) & \text{if } r_0 > r_1. \end{cases}$$

Here $h_\lambda(i,j)$ is the *hook-length* at (i,j) in λ and for $\lambda = (\lambda_1, \lambda_2, \ldots)$ the hook-length $h_\lambda(i,j)$ is defined by

$$h_\lambda(i,j) = \lambda_i - j + |\{\lambda_k; \lambda_k \ge j\}| - i + 1.$$

Since $\lambda^{(i-1)} \underset{1}{\supset} \lambda^{(i)} \underset{1}{\supset} \lambda^{(i+1)}$, we can define the axial distance $d_1 = d(\lambda^{(i+1)}, \lambda^{(i)}, \lambda^{(i-1)})$. Similarly, since $\mu^{(i-1)} \underset{1}{\subset} \mu^{(i)} \underset{1}{\subset} \mu^{(i+1)}$, we can define the axial distance $d_2 = d(\mu^{(i-1)}, \mu^{(i)}, \mu^{(i+1)})$. If $|d_1| \ge 2$ [resp. $|d_2| \ge 2$], then there is a unique Young diagram $\lambda' \neq \lambda$ [resp. $\mu' \neq \mu$] which satisfies $\lambda^{(i-1)} \underset{1}{\supset} \lambda' \underset{1}{\supset} \lambda^{(i+1)}$ [resp. $\mu^{(i-1)} \underset{1}{\subset} \mu' \underset{1}{\subset} \mu^{(i+1)}$]. Let Q_1, Q_2, Q_3 be tableaux of shape $\boldsymbol{\alpha}$ which are obtained from P by replacing $(\lambda^{(i)}, \mu^{(i)})$ on the j-th and the $(j+1)$-st board of $\boldsymbol{\alpha}^{(i)}$ with $(\lambda^{(i)}, \mu')$, $(\lambda', \mu^{(i)})$, (λ', μ') respectively. For the basis elements given by the above tableaux, we define the linear map by the following matrix:

$$(v_P, v_{Q_1}, v_{Q_2}, v_{Q_3}) \longmapsto (v_P, v_{Q_1}, v_{Q_2}, v_{Q_3}) A_i,$$

where

$$A_i = \begin{pmatrix} \frac{1}{d_1 d_2} & \frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} & \sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} & \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} \\ \frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\frac{1}{d_1 d_2} & \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} \\ \sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} & \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\frac{1}{d_1 d_2} & -\frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} \\ \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} & -\frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} & \frac{1}{d_1 d_2} \end{pmatrix}.$$

Next, we consider the case that $\boldsymbol{\alpha}^{(i+1)}$ is obtained from $\boldsymbol{\alpha}^{(i-1)}$ by removing one box from a Young diagram on the j-th board and adding the box to a Young diagram on the $(j+2)$-nd board. Let λ, μ, ν be the Young diagrams on the j-th, the $(j+1)$-st and the $(j+2)$-nd boards of $\boldsymbol{\alpha}^{(i-1)}$ respectively. Then we denote by λ^-, μ, ν^+ the corresponding Young diagrams of $\boldsymbol{\alpha}^{(i+1)}$ respectively. Let $\{\mu(+,k) | k = 1, 2, \ldots, p(\mu)\}$ [resp. $\{\mu(-,k') | k' = 1, 2, \ldots, p(\mu) - 1\}$] be the set of all the Young diagrams which satisfy $\mu(+,k) \underset{1}{\supset} \mu$ [resp. $\mu(-,k') \underset{1}{\subset} \mu$] and let $P_1, P_2, \ldots, P_{p(\mu)}$ [resp. $Q_1, Q_2, \ldots, Q_{p(\mu)-1}$] be all the tableaux which are obtained from P by replacing the Young diagrams on the j-th, the $(j+1)$-st and the $(j+2)$-nd board of $\boldsymbol{\alpha}^{(i)}$ with $\lambda^-, \mu(+,k), \nu$ [resp. $\lambda, \mu(-,k'), \nu^+$]. For the basis elements given by the above tableaux, we define the linear map by the following matrix:

$$\begin{aligned} (A_i)_{P_k, P_{k'}} &= \sqrt{\frac{h(\mu)^2}{h(\mu(+,k))h(\mu(+,k'))}}, \\ (A_i)_{P_k, Q_{k'}} &= (A_i)_{Q_{k'}, P_k} = \frac{1}{d(\mu(-,k'), \mu, \mu(+,k))}\sqrt{\frac{h(\mu)^2}{h(\mu(+,k))h(\mu(-,k'))}}, \\ (A_i)_{Q_k, Q_{k'}} &= 0. \end{aligned}$$

Here $h(\mu)$ is the product of all the hook-lengths in μ:

$$h(\mu) = \prod_{(i,j)\in\mu} h_\mu(i,j).$$

Finally, we consider the remaining cases. In these cases, if $\boldsymbol{\alpha}^{(i)}$ is obtained from $\boldsymbol{\alpha}^{(i-1)}$ by moving one box to the next board and if $\boldsymbol{\alpha}^{(i+1)}$ is obtained from $\boldsymbol{\alpha}^{(i)}$ by moving another box to the next board in a tableau P, then exchanging the i-th step and the $(i+1)$-st step, we have another tableau Q. For the basis elements given by the above tableaux, we define the linear map by the following matrix:

$$(v_P, v_Q) \longmapsto (v_P, v_Q)A_i = (v_P, v_Q)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now we have completed the preparation, we state the following main result.

Theorem 3. *Let $\boldsymbol{\alpha} = [\alpha(1), \ldots, \alpha(n)]$ be an n-tuple of Young diagrams in $\Lambda_n(n)$ and β a non-zero parameter in $\mathbb{C}$.*

(1) *Define $\rho_{\boldsymbol{\alpha}}$ as follows:*

$$\begin{aligned} \rho_{\boldsymbol{\alpha}}(s_i)v_P &= \sum_{Q\in\mathbb{T}(\boldsymbol{\alpha})} (A_i)_{QP} v_Q, \\ \rho_{\boldsymbol{\alpha}}(f)v_P &= \begin{cases} \beta v_P & \text{if } \boldsymbol{\alpha}^{(2)} = [\emptyset, \square, \emptyset, \ldots, \emptyset] \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then $(\rho_{\boldsymbol{\alpha}}, V(\boldsymbol{\alpha}))$ defines an irreducible representation of $A_n(\beta)$.

(2) *For $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \Lambda_n(n)$, the irreducible representations $\rho_{\boldsymbol{\alpha}}$ and $\rho_{\boldsymbol{\alpha}'}$ of $A_n(\beta)$ are equivalent if and only if $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$.*

(3) *Conversely, for any irreducible representation ρ of $A_n(\beta)$, there exists an $\boldsymbol{\alpha} \in \Lambda_n(n)$ such that ρ and $\rho_{\boldsymbol{\alpha}}$ are equivalent.*

Let λ be a Young diagram of size n. Then λ affords an irreducible representation of $\mathfrak{S}_n$. Let f^λ be its degree. Then the degree of $\rho_{\boldsymbol{\alpha}}$ is afforded by the following formula:

$$\deg \rho_{\boldsymbol{\alpha}} = \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_n!} \cdot \frac{f^{\alpha(1)}f^{\alpha(2)}\cdots f^{\alpha(n)}}{\alpha_1!\alpha_2!\cdots\alpha_n!},$$

where $\alpha_j = |\alpha(j)|$ and $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a partition of n which corresponds to $(1^{\alpha_1}, 2^{\alpha_2}, \ldots, n^{\alpha_n})$. Hence

$$\begin{aligned} \sum_{\boldsymbol{\alpha}\in\Lambda_n(n)} (\deg \rho_{\boldsymbol{\alpha}})^2 &= \sum_{\boldsymbol{\alpha}\in\Lambda_n(n)} \left(\frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_n!} \cdot \frac{f^{\alpha(1)}f^{\alpha(2)}\cdots f^{\alpha(n)}}{\alpha_1!\alpha_2!\cdots\alpha_n!}\right)^2 \\ &= \sum_{\lambda\in P(n)} \left(\frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_n!}\right)^2 \cdot \frac{1}{\alpha_1!\alpha_2!\cdots\alpha_n!} \\ &= \dim A_n(\beta). \end{aligned}$$

This implies the following.

Theorem 4. *Let $\Lambda_n(n)$ be the set of n-tuples of Young diagrams of height n. If $\beta \neq 0$, then the party algebra $A_n(\beta)$ is semisimple and $\{\rho_{\boldsymbol{\alpha}}; \boldsymbol{\alpha} \in \Lambda_n(n)\}$ gives a complete set of irreducible representations of $A_n(\beta)$.*

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