QUOTIENT COMPLEXES AND LEXICOGRAPHIC SHELLABILITY (EXTENDED ABSTRACT)

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Abstract. Let $\Pi_{n,k,k}$ and $\Pi_{n,k,h}$, $h < k$, denote the intersection lattices of the k-equal subspace arrangement of type $\mathcal{D}_n$ and the k,h-equal subspace arrangement of type $\mathcal{B}_n$ respectively. Denote by S_n^B the group of signed permutations. We show that $\Delta(\Pi_{n,k,k})/S_n^B$ is collapsible. For $\Delta(\Pi_{n,k,h})/S_n^B$, $h < k$, we show the following. If $n \equiv 0 \pmod{k}$, then it is homotopy equivalent to a sphere of dimension $\frac{2n}{k} - 2$. If $n \equiv h \pmod{k}$, then it is homotopy equivalent to a sphere of dimension $2\frac{n-h}{k} - 1$. Otherwise, it is contractible. Immediate consequences for the multiplicity of the trivial characters in the representations of S_n^B on the homology groups of $\Delta(\Pi_{n,k,k})$ and $\Delta(\Pi_{n,k,h})$ are stated.

In order to establish these results, we are led to introduce a lexicographic shelling condition for a class of not necessarily pure triangulated spaces. It specializes to the CL-shellability of Björner and Wachs [4] when the triangulated space is an order complex of a poset.

1. Introduction

Kozlov [9] studied the triangulated space $\Delta(\Pi_n)/S_n$, i.e. the order complex of the partition lattice modulo the symmetric group, and showed that it is collapsible. The partition lattice occurs in a variety of combinatorial subjects. Of interest here is that it is (isomorphic to) the intersection lattice $L(\mathcal{A}_n)$ of the braid arrangement. This is the arrangement of reflecting hyperplanes of a Coxeter group of type A_{n-1} (for Coxeter group terminology, see Humphreys [8]). In fact, Kozlov used a larger collection of subspace arrangements, including the k-equal braid arrangement $\mathcal{A}_{n,k}$. It seems natural to consider complexes originating from other Coxeter groups.

We determine the homology groups of two families of quotient complexes $\Delta(L(\mathcal{H}))/G$, namely when $\mathcal{H} = \mathcal{D}_{n,k}$, the k-equal subspace arrangement of type D_n, and when $\mathcal{H} = \mathcal{B}_{n,k,h}$, the k,h-equal subspace arrangement of type B_n. In particular, the arrangements of reflecting hyperplanes of Coxeter groups of type B_n and D_n are special cases ($\mathcal{B}_{n,2,1}$ and $\mathcal{D}_{n,2}$ respectively). G in our case is the group of signed permutations, S_n^B, which has a natural action on these lattices.

Now we state the main results. Below, we use the notation $\Pi_{n,k,k} = L(\mathcal{D}_{n,k})$ and $\Pi_{n,k,h} = L(\mathcal{B}_{n,k,h})$. We justify this in Section 2 to which we also refer for all definitions not given here.

Theorem 1.1. *$\Delta(\Pi_{n,k,k})/S_n^B$ is collapsible.*

Theorem 1.2. *Let $h < k$. If $n \equiv 0 \pmod{k}$, then $\Delta(\Pi_{n,k,h})/S_n^B$ is homotopy equivalent to a sphere of dimension $\frac{2n}{k} - 2$. If $n \equiv h \pmod{k}$, then $\Delta(\Pi_{n,k,h})/S_n^B$ is homotopy equivalent to a sphere of dimension $2\frac{n-h}{k} - 1$. Otherwise, $\Delta(\Pi_{n,k,h})/S_n^B$ is contractible.*

Using standard facts we obtain the following corollary concerning representations of S_n^B.

Corollary 1.3. *Let K be a field with $char(K) \notin [n]$. Then*

1. *The trivial character of the induced representation of S_n^B on the vector space $\widetilde{H}_i(\Delta(\Pi_{n,k,k})), K)$ has multiplicity zero for all i.*
2. *Let $h < k$. The trivial character of the induced representation of S_n^B on the vector space $\widetilde{H}_i(\Delta(\Pi_{n,k,h}), K)$ has multiplicity one if $n \equiv 0 \pmod{k}$ and $i = \frac{2n}{k} - 2$ or if $n \equiv h \pmod{k}$ and $i = 2\frac{n-h}{k} - 1$. Otherwise it has multiplicity zero.*

In the proofs of both Theorem 1.1 and Theorem 1.2 we use an acyclic matching technique which is briefly discussed in Section 2. This immediately yields Theorem 1.1. To prove Theorem 1.2, however, we need to prove shellability of a certain subcomplex of $\Delta(\Pi_{n,k,h})/S_n^B$. To do this, we are led to introduce a lexicographic shellability condition for a class of typically non-pure triangulated spaces. In Section 3 we describe this condition. It coincides with the CL-shellability of Björner and Wachs [4] when restricted to order complexes of posets. A similar condition for pure, balanced triangulated spaces was recently introduced by Hersh [7]. In order to apply the shellability condition we need a combinatorial description of our triangulated spaces in terms of trees. This description is given in Section 4.

Acknowledgement. The author is indebted to Dmitry Kozlov for suggesting the problem and for providing valuable comments. The author is also grateful to the anonymous referee for his/her suggestions.

2. Basic definitions and notation

In this section we collect basic definitions and agree on notation. For anything not explained here, we refer to the standard textbooks by Stanley [12] (combinatorics) and Munkres [11] (topology).

2.1. **Shelling triangulated spaces.** We define a *triangulated space* to be a regular cell complex (see e.g. Björner [3, Section 12]) whose face poset has the property that every lower interval is a boolean algebra. Hence, a triangulated space is almost a simplicial complex, except that several simplices may share the same vertex set. Hersh [7] uses the term *boolean cell complex.*

Björner and Wachs [5, Def. 13.1] define shellability for an arbitrary regular cell complex. Translation to the case of triangulable spaces yields the following plausible definition.

Definition 2.1. *An ordering $F_1, \dots, F_t$ of the facets (inclusion-maximal simplices) of a triangulated space Δ is a* shelling order *of Δ if $F_j \cap (\cup_{\alpha<j} F_\alpha)$ has codimension 1 in F_j for all $2 \le j \le t$. If there exists a shelling order of Δ, then Δ is* shellable.

We think of a shelling order as a way of putting together Δ facet by facet. Therefore we say that F_j *attaches* over $F_j \cap (\cup_{\alpha<j} F_\alpha)$.

2.2. **Discrete Morse theory.** Let Δ be a regular cell complex, although for our purposes it would suffice to consider triangulated spaces. A *matching* on the face poset $P(\Delta)$ is a partition of $P(\Delta)$ into three sets X, Y and Z, such that there exists a bijection $\phi : Y \to X$ with the property that y is covered by $\phi(y)$ for all $y \in Y$. Z is called the *critical set* of the matching. The matching is *acyclic* if there exists no sequence $y_1, \dots, y_q \in Y$ such that $y_q = y_1$, $y_i \ne y_{i+1}$ and $\phi(y_i)$ covers y_{i+1} for all $i \in [q-1]$.

From Forman's work [6], the next result follows. See also Kozlov [9] for a direct combinatorial proof.

Theorem 2.2. *Suppose we have an acyclic matching on $P(\Delta)$ with critical set Z. If Z is a subcomplex of Δ, then Z can be obtained from Δ by a sequence of elementary collapses. In particular, Z and Δ are homotopy equivalent.*

2.3. BD**-arrangements and quotient complexes.** Throughout the rest of the paper we will frequently encounter the triple (n, k, h). Whenever these integers appear, it will be assumed that $1 \le h \le k \le n$ if nothing else is explicitly stated.

Definition 2.3. *$\mathcal{D}_{n,k}$,* the k-equal subspace arrangement of type D_n, *is the collection of all linear subspaces of the form*

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \tau_1 x_{i_1} = \cdots = \tau_k x_{i_k}\}$$

for $1 \le i_1 < \cdots < i_k \le n$ and $\tau_i \in \{-1, 1\}$ for all i.

Definition 2.4. *For $h < k$, we define $\mathcal{B}_{n,k,h}$, the k,h-equal subspace arrangement of type B_n, to be the union of $\mathcal{D}_{n,k}$ and the collection of linear subspaces of the form*

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_{i_1} = \cdots = x_{i_h} = 0\}$$

for $1 \leq i_1 < \cdots < i_h \leq n$.

The special cases $\mathcal{B}_n = \mathcal{B}_{n,2,1}$ and $\mathcal{D}_n = \mathcal{D}_{n,2}$ are the ordinary *hyperplane arrangements of types B_n and D_n* respectively.

Zaslavsky [14] has given a nice description of the intersection lattices $L(\mathcal{B}_{n,k,h})$ and $L(\mathcal{D}_{n,k})$ in terms of lattices of signed graphs. We will, however, only briefly consider the structure of these lattices, so we settle for a more naive description of $L(\mathcal{B}_{n,k,h})$ and $L(\mathcal{D}_{n,k})$. For more on subspace arrangements and intersection lattices we refer to Björner [2].

Definition 2.5. *Let $\Pi_{n,k,h}$ be the lattice of set partitions of the set $\{-1, 1, -2, 2, \dots, -n, n\}$ such that the following conditions hold:*

1. *The partitions are sign-symmetric, i.e. if all plus and minus signs are interchanged, then the partition is unchanged.*
2. *There is at most one self-symmetric block, i.e. block containing both $-a$ and a for some $a \in [n] := \{1, \dots, n\}$.*
3. *The non-singleton non-self-symmetric blocks have size at least k.*
4. *The self-symmetric block has size at least $2h$, if it exists.*

With the obvious interpretation $\tau_s x_s = \tau_t x_t$ iff $\tau_s s$ and $\tau_t t$ are in the same block, we see that $\Pi_{n,k,h}$, $h < k$, is isomorphic to $L(\mathcal{B}_{n,k,h})$ and that $\Pi_{n,k,k}$ is isomorphic to $L(\mathcal{D}_{n,k})$.

Throughout we will assume that all posets we consider are finite. We will not make any notational distinctions between a simplicial complex and its geometric realization. Given a poset P equipped with a maximal element $\hat{1}$ and a minimal element $\hat{0}$, we let $\overline{P}$ denote the *proper part* $P \setminus \{\hat{0}, \hat{1}\}$. The *order complex* $\Delta(P)$ is the simplicial complex having the chains of $\overline{P}$ as simplices. If G is a group acting on P in an order-preserving way, we may define $\Delta(P)/G$ as the triangulated space of G-orbits of simplices of $\Delta(P)$. In general, $\Delta(P)/G$ is not a simplicial complex, since there may be more than one simplex on the same set of vertices. Babson and Kozlov [1] give conditions that guarantee $\Delta(P)/G \cong \Delta(P/G)$. Earlier, Welker [13] had given specific examples of posets and groups with this property.

To shorten notation, we introduce $\Delta_{n,k,h} := \Delta(\Pi_{n,k,h})/S_n^B$. Here, S_n^B is the group of signed permutations, which acts in a natural way on $\Pi_{n,k,h}$. It is a Coxeter group of type B_n. For our purposes, it suffices to view S_n^B as the group of permutations π of the set $\{-1, 1, -2, 2, \dots, -n, n\}$ such that $\pi(a) = -\pi(-a)$ for all $a \in [n]$.

A more thorough analysis of the simplex structure of $\Delta_{n,k,h}$ takes place in Section 4. Here we will only describe the vertices, $vert(\Delta_{n,k,h})$, of $\Delta_{n,k,h}$. Given two partitions $\pi, \tau \in \Pi_{n,k,h}$, it is clear that there exists a $g \in S_n^B$ such that $g\tau = \pi$ if and only if there is a bijection between the blocks of τ and the blocks of π which respects block size and commutes with the operation of interchanging all plus and minus signs.

Let N_n denote the set of integer partitions of n in which we allow at most one part, the *null* part, to be distinguished. Define the *nullity*, $null(\lambda)$, of λ to be the size of the null part of λ, or zero, if λ has no null part. Order N_n by the rule: $\lambda \leq \kappa$ if λ refines κ as an integer partition and $null(\lambda) \leq null(\kappa)$.

Definition 2.6. *We let $N_{n,k,h}$ denote the subposet of N_n induced by the elements λ with $null(\lambda) \notin [h-1]$ in which all non-null parts are either singletons or have size at least k.*

Consider the map $type : vert(\Delta_{n,k,h}) \to \overline{N_{n,k,h}}$ defined as follows. Pick $v \in vert(\Delta_{n,k,h})$ and $\tau_v \in \Pi_{n,k,h}$ such that $orb(\tau_v) = v$, where $orb()$ denotes S_n^B-orbit. A size $2s$ self-symmetric block in τ_v gives rise to a size s null part in $type(v)$. The other blocks in τ_v occur in sign-reflected pairs.

Each such pair of blocks of size s gives rise to a size s (non-null) part in $type(v)$. For example, with $(n,k,h) = (4,2,1)$, we have $type(orb(\{\{-1,1\},\{-2,3\},\{4\},\{2,-3\},\{-4\}\})) = \{\bar{1},2,1\}$, where the bar indicates null part. Obviously, in the light of the discussion above, *type* is well-defined and bijective. Hereafter, we will consider the vertices of $\Delta_{n,k,h}$ to be elements of $N_{n,k,h}$.

3. Lexicographic shellings of O-complexes

A d-dimensional triangulated space Δ is *balanced* if there exists a map $f : vert(\Delta) \to [d+1]$ whose restrictions to all simplices are injective. Δ is *pure* if all facets are of dimension d. Hersh [7] gave a lexicographic shelling condition for pure balanced triangulated spaces. In this section we modify her ideas and give a lexicographic shelling condition for a class of possibly non-pure triangulated spaces.

Definition 3.1. *An* ordered complex*, or* O-complex*, is a pair (Δ, P), where Δ is a triangulated space and P a poset on the ground set $vert(\Delta)$ such that the vertex set of every simplex in Δ is a P-chain and every P-chain is the vertex set of at least one simplex in Δ.*

Remark. An O-complex (Δ, P) is a balanced triangulated space (define $f(v)$ to be the maximal cardinality of any P-chain with v as maximal element). If we consider all balanced triangulated spaces instead of just O-complexes, then everything in this section generalizes without much effort. However, we choose not to adopt this approach, since the notation tends to get messy. For our applications, O-complexes suffice.

Natural examples of O-complexes are quotient order complexes $\Delta(P)/G$. In this case, the O-complex is the pair $(\Delta(P)/G, \Pi)$, where Π is the induced poset on the G-orbits of P's elements.

When speaking of topological properties of an O-complex (Δ, P), we always have the properties of Δ in mind.

Let $\Delta = (\Delta, P)$ be an O-complex. We define $\hat{P} := P \cup \{\hat{0}, \hat{1}\}$, where $\hat{0} < p < \hat{1}$ for all $p \in P$. $\hat{\Delta}$ denotes the triangulated space whose facets are $\{F \cup \{\hat{0}, \hat{1}\} \mid F \text{ is a facet of } \Delta\}$. This is the *join* of Δ and the one-dimensional simplex on $\{\hat{0}, \hat{1}\}$ (see e.g. Björner [3, Section 9]).

A *root simplex* of $\hat{\Delta}$ is a simplex whose vertex set comprises a maximal chain in some lower interval $[\hat{0}, v]$, $v \neq \hat{0}$, of $\hat{P}$. Furthermore, a simplex c with vertex set of the form $\{\hat{0} \to \ldots \to u < v\}$, $u \not\to v$, determines the *rooted interval* consisting of all root simplices containing c and having vertex set in $[\hat{0}, v]$. (Here, $\to$ is the covering relation of $\hat{P}$.)

Let $R(\hat{\Delta})$ be the set of root simplices of $\hat{\Delta}$. A *chain labelling* of Δ is a map $\lambda : R(\hat{\Delta}) \to \Lambda$, where Λ is some poset of labels.

We pause for a moment to help the intuition. Suppose c is a root simplex of $\hat{\Delta}$ with $vert(c) = \{\hat{0} \to v_1 \to \ldots \to v_i\}$. Unless $i = 1$ or $v_i = \hat{1}$, we can think of the label $\lambda(c)$ as being assigned to the edge in Δ between v_{i-1} and v_i which is contained in c. If $i = 1$ or $v_i = \hat{1}$, then this edge is not in Δ, but we may view $\lambda(c)$ as labelling the vertex v_1 (if $i = 1$) or v_{i-1} (if $v_i = \hat{1}$). In all cases, the label is possibly dependent on the simplex c containing the edge (or the vertex). Note that if Δ is the (simplicial) order complex of the poset P, then a chain labelling of Δ is nothing but an ordinary chain-edge labelling of $\hat{P}$ (see Björner and Wachs [4, Def. 5.1]).

Consider a simplex c with vertices $vert(c) = \{v_0 < \cdots < v_r\}$. To shorten notation, we let $c^{i \to j}$, $-1 \leq i < j \leq r+1$, denote the unique simplex contained in c on vertex set $\{v_0, \ldots, v_i, v_j, \ldots, v_r\}$. We also let $c^{i \to} := c^{i \to r+1}$, $c^i := c^{i-1 \to i+1}$ and $c^{i_1 \to j_1, \ldots, i_m \to j_m} := (\ldots(c^{i_m \to j_m})\ldots)^{i_1 \to j_1}$.

Given a chain labelling λ, we define the *descent set* $D(c)$ of a root simplex c with $vert(c) = \{\hat{0} \to v_1 \to \ldots \to v_r\}$ to be $D(c) = \{i \in [r-1] \mid \lambda(c^{i \to}) \not\leq \lambda(c^{i+1 \to})\}$. Consider the rooted interval given by the simplex $c^{i \to j, j \to}$. We say that c is *falling* on the interval if $D(c) \supseteq \{i+1, \ldots, j-1\}$. c is *rising* on the interval if, instead, $D(c) \cap \{i+1, \ldots, j-1\} = \emptyset$. We compare two root simplices b and c on the rooted interval using the lexicographical order: $b <_{lex} c$ iff $\lambda(b^{t \to}) < \lambda(c^{t \to})$, where t is the smallest index such that $i < t \leq j$ and $\lambda(b^{t \to}) \neq \lambda(c^{t \to})$.

We now give a lexicographic shelling condition for O-complexes.

Definition 3.2. *An O-complex Δ is* lexicographically shellable *if there exists a chain labelling of Δ such that the following four conditions are fulfilled:*

(1) *Every rooted interval contains a unique simplex which is rising on the interval.*
(2) *In every rooted interval, the rising simplex is lexicographically smaller than all other simplices.*
(3) *Let $F_1, \dots, F_t$ be an ordering of the facets of $\hat{\Delta}$ which is a linear extension of the lexicographical order. Let c be a maximal simplex in $F_p \cap F_r$, $p < r$. Write $c = F_r^{s_1 \to t_1, \dots, s_m \to t_m}$, where $s_1 < t_1 < \cdots < s_m < t_m$. Let $j \in [m]$ be maximal such that F_r is rising on all rooted intervals given by $F_r^{s_i \to t_i, t_i \to}$, $i < j$. Then, for some $q < r$ and $i \leq j$, we have $F_r^{s_i \to t_i} \subseteq F_q \cap F_r$.*
(4) *Let c be a maximal simplex in $F_q \cap (\cup_{\alpha < q} F_\alpha)$. Suppose $c = F_q^{s \to t}$ and that F_q is rising on the rooted interval given by $F_q^{s \to t, t \to}$. Then $codim_{F_q}(c) = 1$.*

Remark. If Δ is the (simplicial) order complex of the poset P, then Condition 3 is trivially satisfied and Condition 4 follows from Conditions 1 and 2. Hence, in this case Definition 3.2 defines ordinary CL-shellability (see Björner and Wachs [4, Def. 5.2]) for $\hat{P}$. Then Condition 3 only reflects the fact, that if two maximal P-chains $c_1 <_{lex} c_2$ differ on several intervals, then one can find a chain $d <_{lex} c_2$ which only differs from c_2 on the first of those intervals (simply by letting the first part of d imitate c_1 and the second part c_2). Condition 4 follows since Conditions 1 and 2 imply that $F_1, \dots, F_t$ is a shelling order.

Theorem 3.3. *If an O-complex Δ is lexicographically shellable, then it is shellable.*

Proposition 3.4. *Let G be a group acting on the poset P in an order preserving way. Then $\Delta = \Delta(P)/G$ is lexicographically shellable if and only if it has a chain labelling satisfying Conditions 1 and 2 of Definition 3.2 together with the following condition:*

$3'$. *Let $F_1, \dots, F_t$ be an ordering of the facets of $\hat{\Delta}$ which is a linear extension of the lexicographical order. Let c be a simplex in $F_r \cap (\cup_{\alpha < r} F_\alpha)$. Write $c = F_r^{s_1 \to t_1, \dots, s_m \to t_m}$, where $s_1 < t_1 < \cdots < s_m < t_m$. Suppose F_r is rising on all rooted intervals given by $F_r^{s_i \to t_i, t_i \to}$. Then $c \subseteq b \subseteq F_r \cap (\cup_{\alpha < r} F_\alpha)$ for some simplex b with $codim_{F_r}(b) = 1$.*

Remark. As with simplicial complexes, if an O-complex Δ is shellable, then it is homotopy equivalent to a wedge of spheres. Just as in lexicographic shellings of posets, the falling facets correspond to simplices attached over their entire boundaries. However, unlike the ordinary case, there may exist other simplices that attach over their entire boundaries. Consider, e.g., two facets, F and G, on the same vertex set. Even if, say, $1 \notin D(F) \cup D(G)$, we may still have $F^1 = G^1$. Hence, the last attached of F and G will be attached over the boundary simplex $F^1 = G^1$ even though 1 is not a descent. This fact seems to have been overlooked in [7, Rem. 2.4]. We have the following (weak) proposition:

Proposition 3.5. *If an O-complex Δ is lexicographically shellable, it is homotopy equivalent to a wedge of spheres. Its (reduced) Betti numbers satisfy*

$\widetilde{\beta}_i(\Delta) \geq$ *#falling facets on $i+1$ vertices.*

4. Describing $\Delta_{n,k,h}$ using trees

4.1. (n, k, h)**-trees.** In this section we give a description of the simplices of $\Delta_{n,k,h}$ in terms of a certain kind of trees. With modifications, we follow Kozlov's [10, Section 4] description of some complexes related to the order complex of the partition lattice modulo the symmetric group.

In the following, we will suppose that all trees are finite. Given a tree T, let $V(T)$ denote the vertex set of T. For a rooted tree T, let $l_i(T)$ be the number of vertices on distance i from the root. A rooted tree T is called a *graded tree of rank* r if the distance from an arbitrary leaf to the root is $r+1$ and $1 = l_0(T) < l_1(T) < \cdots < l_{r+1}(T)$. In such a tree, the *depth* of a vertex v is the distance from the root to v.

Let $<\overline{n}>$ denote the set $\{\overline{0}, \overline{1}, \dots, \overline{n}\}$. Define $\overline{k} + l := \overline{k+l}$ for integers k and l, and extend the definition by associativity and commutativity. (Sums of the type $\overline{k} + \overline{l}$ are not defined.)

Definition 4.1. *An (n, k, h)-tree of rank r is a pair (T, η), where T is a graded tree of rank r and $\eta : V(T) \to \{\overline{0}, \overline{h}, \overline{h+1}, \dots, \overline{n}\} \cup \{1, k, k+1, \dots, n\}$ is a labelling of T:s vertices such that*

(1) *$\eta(\rho) = \overline{n}$, where ρ is the root of T.*
(2) *For all non-leaf vertices $v \in V(T)$, we have $\eta(v) = \sum \eta(w)$ (sum over all children w of v).*
(3) *On every depth, there is at least one non-trivial vertex v, i.e. $\eta(v) \notin \{\overline{0}, 1\}$.*

We often abuse notation and write $T = (T, \eta)$.

Remark. In particular, the second condition implies that if $\eta(v) \in <\overline{n}>$ for a non-leaf v, then $\eta(w) \in <\overline{n}>$ for exactly one child w of v.

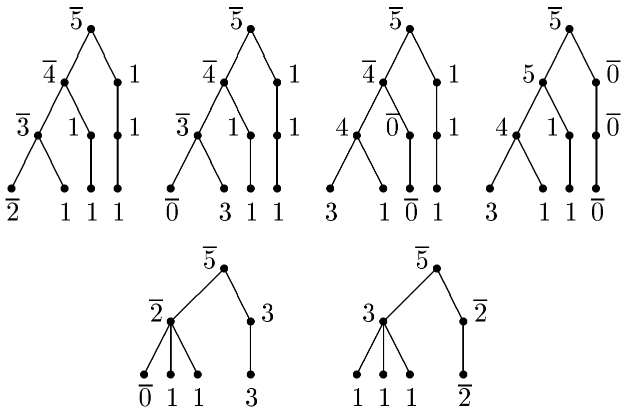

FIGURE 1. *All maximal $(5, 3, 2)$-trees. The upper ones are of rank 2 and the lower of rank 1.*

Let $T^r_{n,k,h}$ denote the set of (n, k, h)-trees of rank r (we will not distinguish between isomorphic trees), and let $T_{n,k,h}$ be the set of all (n, k, h)-trees so that, in particular, $T_{n,k,h} = \cup_r T^r_{n,k,h}$.

For an integer i, $0 < i \leq r + 1$, we define the deletion operator $\delta^i : T^r_{n,k,h} \to T^{r-1}_{n,k,h}$. $\delta^i(T) = \delta^i((T, \eta))$ is defined as (T^i, η^i), where T^i is the tree obtained from T by deleting all vertices of depth i and letting the grandchildren of the depth $i - 1$ vertices be their new children. η^i is the restriction of η to $V(T^i)$. For convenience, we introduce $\delta^{i \to j} := \delta^{i+1} \circ \delta^{i+2} \circ \dots \circ \delta^{j-1}$ for $i > j + 1$. We will also use the notation $\delta^{i\to} := \delta^{i+1} \circ \delta^{i+2} \circ \dots \circ \delta^{r+1}$.

4.2. **A description of $\Delta_{n,k,h}$.** Recall from Section 2.3 the description of the vertices of $\Delta_{n,k,h}$ in terms of the elements of $N_{n,k,h}$. Note that each level, i.e. set of vertices on some fixed depth, of $T \in T_{n,k,h}$ can be viewed as an element $\lambda \in N_{n,k,h}$. The (labelled) vertices of T correspond to parts in λ. A vertex v such that $\eta(v) \in <\overline{n}>$ corresponds to a null part. (We interpret $\eta(v) = \overline{0}$ as the non-existence of a null part.)

Let $\Delta^r_{n,k,h}$ be the set of r-dimensional simplices in $\Delta_{n,k,h}$. We describe a mapping $\psi : \Delta^r_{n,k,h} \to T^r_{n,k,h}$ as follows:

Take a simplex $c \in \Delta^r_{n,k,h}$ and let the chain $\pi = \{\pi_{r+1} < \pi_r \cdots < \pi_1\} \subseteq \Pi_{n,k,h}$ be a representative of c. We construct $\psi(c)$ level by level. The (labelled) vertices on depth i of $\psi(c)$ are the parts in $orb(\pi_i) \in N_{n,k,h}$. We put an edge between two vertices a and b, of depths i and $i+1$ respectively, if the block of π_i corresponding to a is refined in π by the block of π_{i+1} corresponding to b. Finally, we add the root $\overline{n}$ and put edges from it to all vertices of depth 1. It follows immediately from the construction, that ψ is well-defined, i.e. $\psi(c)$ does not depend on the choice of representative of c.

The following result is analogous to [10, Thm. 4.4]. It allows us to use $T_{n,k,h}$ as a model of $\Delta_{n,k,h}$.

Proposition 4.2. *The mapping $\psi : \Delta^r_{n,k,h} \to T^r_{n,k,h}$ is bijective. Furthermore, under ψ, inclusion in $\Delta_{n,k,h}$ corresponds to deletion in $T_{n,k,h}$. In other words, for two simplices $c_1, c_2 \in \Delta_{n,k,h}$ we have $c_1 \subseteq c_2$ iff $\psi(c_1) = \delta^{i_1} \circ \cdots \circ \delta^{i_t}(\psi(c_2))$ for some $i_1, \ldots, i_t$.*

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