SOME PLANAR ALGEBRAS RELATED TO GRAPHS

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ABSTRACT. A planar algebra consists of a collection of vector spaces which is closed under certain operators defined by diagrams known as planar tangles. Although planar algebras were introduced to study problems in subfactors, they have applications in combinatorics. We shall discuss planar algebras as a framework for graph rewriting systems similar to the skein theory of knots. We focus on the prototypical example: the Reidemeister moves and spin models for link invariants. We discuss the planar algebras related to Higman-Sims graph and the Petersen graph. Despite many similarities, the former admits Reidemeister-like reductions while the latter does not.

RÉSUMÉ. Une algèbre plainaire est une collection d'espaces vectoriels qui est fermée sous certains opérateurs définis par des diagrammes qu'on appelle "enchevêtrements planaires". Bien que les algèbres planaires aient été introduites pour étudier des problém des sous-facteurs, elles ont des applications en combinatoire. Nous discuterons des algèbres planaires comme une structure pour réécrire les graphes, semblables à la théorie de skein des noeuds. Nous nous concentrons sur l'exemple prototypique: les mouvements de Reidemeister et les modèles à spins. Nous discutons des algèbres plainaires produites par les graphes de Higman-Sims et de Petersen. En dépit de beaucoup de propriétés communes, Higman-Sims admet des réductions comme les mouvements de Reidemeister alors que le Petersen n'en admet pas.

1. INTRODUCTION

Planar algebras were introduced to study subfactors [12]. True to their operator algebra origins, the study of planar algebras emphasizes the interactions of certain operators. These operators are defined diagrammatically by objects known as planar tangles, and their study has a knot theoretic flavor, very much like Conway's tangles and skein relations [3]. This approach is strongly influenced by the deep connections between knot theory and subfactors [10, 9]. No knowledge of subfactors is required here, but we note that some of the technical conditions in the definition of a planar algebra arise from subfactor considerations.

Our goal here is to discuss some combinatorial aspects and applications of planar algebras. In particular, we discuss the use of planar algebras as a framework for a skein theoretic approach to graph rewriting. To illustrate this we shall focus on the prototypical example: the Reidemeister moves and spin models for link invariants. We also compare particular instances related to the Higman-Sims graph and the Petersen graph. Despite many similarities, the former admits Reidemeister-like reductions while the latter does not. We conclude with a discussion of a result which is helpful in studying the skein theory of planar algebras related to graphs.

2. PLANAR ALGEBRAS: DEFINITIONS AND EXAMPLES

2.1. **The planar operad.** A planar tangle T consist of the following collection of objects:

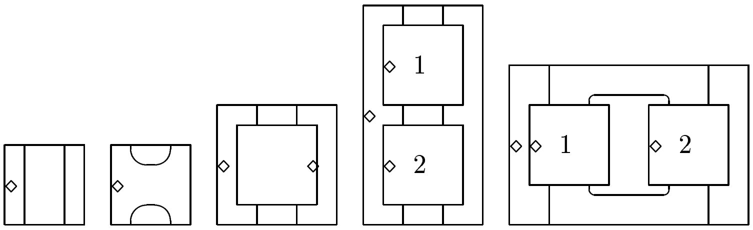
(i) the unit disk D_0;
(ii) some disjoint disks $D_1, \ldots, D_n$ in the interior of D_0;
(iii) a finite number of disjoint smooth curves ("strings") all of whose endpoints lie on the boundary of some disk but which are otherwise interior to $D_0 \setminus \cup_{i=1}^{n} D_i$ such that an even number of strings touch each disk, say $2k_i$ touching D_i (there may be closed loops which touch no disk);
(iv) a checkerboard shading of the regions interior to $D_0 \setminus \cup_{i=1}^{n} D_i$ formed by the strings;

(v) a choice on the boundary of each disk of a privileged interval among those formed by string endpoints and which form part of the boundary for an unshaded region.

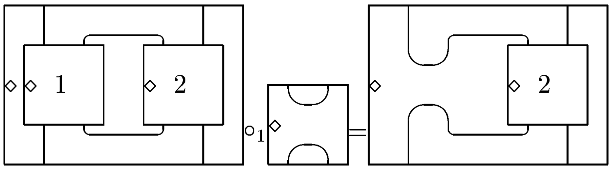
(vi) If some $k_i = 0$, we set k_i to $+$ or $-$ according to whether region near the boundary of D_i is shaded or not. In the former case, the single interval on the boundary of D_i is privileged and in the latter case it is not.

The smooth isotopy class of such an object is called a *planar k_0-tangle*. The *planar operad* is the set of planar tangles with the following composition: An internal disk of a planar tangle can be replaced by a planar tangle by deforming the latter, identifying their privileged intervals and successive string endpoints (provided that the number of string endpoints are compatible), and removing disks being glued together.

We shall use the fact that planar tangles are defined only up to smooth isotopy and draw them as boxes. The privileged interval will be marked with $\diamond$. Furthermore, the shading is uniquely determined by the choice of a privileged interval. We shall number the boxes when necessary to determine an order of operations. Some interesting planar tangles include:



Observe that any of these tangles may be composed into any of the boxes of the last three. Composing the second tangle into the first box of the rightmost tangle gives



2.2. **Planar algebras.** A *(general) planar algebra* is a graded collection of vector spaces $\mathcal{V}_i$ ($i \in \{+,-\} \cup \{1,2,3,\ldots,\}$) over some field $\mathbb{K}$ such that each planar tangle T defines a multilinear map $\otimes \mathcal{V}_{k_i} \to \mathcal{V}_{k_0}$, where the k_i are as in the definition of a planar tangle T. Furthermore, this multilinear map must have a natural homomorphism property. Given planar tangles T_1, T_2, T_3 which admit compositions of T_2 into T_1 and T_3 into this composition, the net result of these compositions in the planar operad does not depend upon the order in which they are carried out: The same must be true for the corresponding multilinear maps on the planar algebra. We also impose a condition on $\mathcal{V}_+$ and $\mathcal{V}_-$. We view these two vector spaces as corresponding to the two shadings of any planar 0-tangle–$\mathcal{V}_+$ to those with a black shading next to the unit disk and $\mathcal{V}_-$ to those with a white shading next to the unit disk. Observe that surrounding the interior component with a closed string reverses its shading. We require that surrounding the interior of a planar 0-tangle with two closed strings yields a multilinear map to $\mathcal{V}_+$ or $\mathcal{V}_-$ which differs from the original multilinear map only by a scalar multiple which depends only upon the shading near the unit circle.

An important statistic of a planar algebra $\cup\mathcal{V}_i$ is the dimension of each vector space $\mathcal{V}_i$, which one could encode in a Hilbert series. In general, the computation of this dimension is an undecidable problem. We shall give give formulas for the dimensions of the component vector spaces of some combinatorial planar algebras we meet certain conditions.

2.3. **The planar algebra of functions on a finite set.** We are interested in planar subalgebras of the following planar algebra. Fix a field $\mathbb{K}$, and let X denote a finite non-empty set. For each positive integer i, let $\mathcal{X}_k$ be the vector space of all functions from X^k to $\mathbb{K}$ ($k > 0$), with $\mathcal{X}_+$, $\mathcal{X}_-$ identified with $\mathbb{K}$. Then $\cup\mathcal{X}_k$ is a planar algebra, where a planar tangle T with the above notation defines a multilinear map $\bigotimes_{i=1}^{n} \mathcal{X}_{k_i} \to \mathcal{X}_{k_0}$ as follows. Index the black regions of T by 1, 2, ..., m. For all i $(0 \le i \le n)$ and for all j $(1 \le j \le k_i)$, let S_{ij} be the index of the j^{th} shaded region incident with D_i when traversing the boundary of D_i clockwise so that the privileged interval is traversed last. Given $f_i \in \mathcal{X}_{k_i}$ $(1 \le i \le n)$, define a function $Z_T : X^{k_0} \to \mathbb{C}$ which, when evaluated at $(x_1, x_2, \ldots, x_{k_0})$, returns

$$\sum\nolimits_{\sigma} \prod\nolimits_{i=1}^{n} f_i(\sigma(S_{i1}), \sigma(S_{i2}), \ldots, \sigma(S_{ik_i})), \tag{1}$$

where σ runs over all maps from $\{1, 2, \ldots, m\}$ to X with $\sigma(S_{0j}) = x_j$.

By definition, Z_T is a multilinear map $\bigotimes_{i=1}^{n} \mathcal{X}_i \to \mathcal{X}_{k_0}$. The homomorphism property of planar algebras follows since the function only depends upon the incidences of the shaded regions and composition merges regions with the same shading. Enclosing a planar 0-tangle with two closed strings preserves the shading of the interior (reveres it twice) but adds an isolated shaded band. Thus this modified tangle corresponds to a multilinear map which is $|X|$ times the multilinear map corresponding to the original tangle. Thus $\cup \mathcal{X}_k$ is a planar algebra.

To understand this action, let us consider the partition function of the left-most planar tangle above. It has a single shaded region, indexed 1. There are no internal disks, so the product is empty, and thus equal to 1. Traversing the boundary of the outside disk, we find $S_{01} = 1$ and $S_{02} = 1$. There is a unique map σ mapping $\{1\}$ to X with $\sigma(1) = x_1$ and $\sigma(1) = x_2$. Thus this planar tangle corresponds to the function $f(x_1, x_2) = \delta_{x_1, x_2}$. We may restate this using matrices. Let M_X denote the set of matrices with entries in $\mathbb{K}$ whose rows and columns are indexed by X. By identifying $\mathcal{X}_2$ with M_X, we see that the planar tangles above represent the identity matrix, the all one's matrix, matrix transposition, matrix multiplication, and entry-wise multiplication of matrices. (A matrix algebra which is closed under these operations is called a *coherent algebra*, a familiar object in algebraic combinatorics).

2.4. **The planar algebra of fixed points for a finite group action.** Let $\mathfrak{G}$ denote a permutation group on X, and for all $i \in \{+, -\} \cup \{1, 2, \ldots\}$, let $\mathcal{F}_i(\mathfrak{G}, X)$ denote the vector subspace of $\mathcal{X}_i$ consisting of those functions which depend only upon the $\mathfrak{G}$-orbit of their inputs. Together these vector spaces form a planar algebra $\mathcal{F}(\mathfrak{G}, X)$, called the *planar algebra of fixed-point for $\mathfrak{G}$ acting on X*. Each vector space $\mathcal{F}_i(\mathfrak{G}, X)$ has dimension equal to the number of $\mathfrak{G}$-orbits of X^i. In particular, it is finite dimensional, and its dimension can be computed via the Cauchy-Frobenius-Burnside formula for the characters of the group $\mathfrak{G}$. Recall that the *permutation representation* of $\mathfrak{G}$ acting on X is the map $\mathfrak{g} \mapsto R(\mathfrak{g}) \in \mathcal{M}_X(\mathbb{C})$ with (x, y)-entry equal to 1 if $y = \mathfrak{g} \cdot x$ and 0 otherwise $(x, y \in X)$. The *permutation character* π of $\mathfrak{G}$ is the function $\pi : \mathfrak{G} \to \mathbb{C}$ given by $\pi(\mathfrak{g}) = \operatorname{Tr} R(\mathfrak{g})$ (the permutation character returns the number of elements of X fixed by g). For all positive integers i, the number $\mathfrak{G}$- orbits of X^i is

$$\dim \mathcal{F}_i(\mathfrak{G}, X) = \frac{1}{|\mathfrak{G}|} \sum_{\mathfrak{g} \in \mathfrak{G}} (\pi(\mathfrak{g}))^i. \tag{2}$$

Below, we shall fix a matrix $W \in M_X$. In this case we shall write $\mathcal{F}^W$ rather that $\mathcal{F}(\mathfrak{Aut}W, X)$, where $\mathfrak{Aut}(W)$ is the set of permutation of X which leave W invariant when it simultaneously permutes the rows and columns of W.

2.5. **Finitely generated planar algebras.** Our interest primarily lies with finitely generated planar subalgebras of $\cup \mathcal{X}_i$. To describe this notion we use labeled planar tangles. Let L_i $(i \in \{+, -\} \cup \{1, 2, \ldots\})$ denote mutually disjoint sets of "labels", and let $L = \cup L_i$. To each label $\ell \in L_i$, associate a function $f_\ell \in \mathcal{X}_i$. Then the *planar algebra $\mathcal{P}^\Omega$ generated by $\Omega = \{f_\ell \mid \ell \in L\}$* is the span of all functions Z_T^Ω defined by (1) as T runs over all planar tangles in which every interior disk D_i of T has a label $\ell_i \in L_{k_i}$.

Observe that $\mathcal{P}_i^\Omega$ is finite dimensional. In fact, $\dim \mathcal{P}_i^\Omega \le \dim \mathcal{F}_i(\mathfrak{G}, X)$, where $\mathfrak{G}$ is the subgroup of all permutations σ on X satisfying $f(\sigma(\vec{x})) = f(\vec{x})$ for all $f \in \Omega$ and for all $\vec{x} \in X^i$ when $f \in \mathcal{X}_i$. It is natural to ask when equality holds in this bound.

For simplicity, we shall consider the planar algebra $\mathcal{P}^W$ generated by a single element of $W \in \mathcal{X}_2$ (a matrix). We are especially interested in the case that W is the adjacency matrix of a graph. One of our main results is a test to decide when $\mathcal{P}^W = \mathcal{F}(\mathfrak{Aut}(W), X)$, allowing us to compute the dimensions of the component vector spaces of $\mathcal{P}^W$. We now describe one more planar algebra which shows to interpret $\mathcal{P}^W$ in terms of graph rewriting systems.

2.6. **The open graph planar algebra.** We extend the planar algebra $\mathcal{P}^W$ by a graph theoretic construction. For simplicity, we shall not consider edge-labelings. By a *multi-digraph*, we mean a pair $\Delta = (V, E)$, where V (the *vertex set*) is a nonempty set and E (the *edge multiset*) is a multiset of ordered pairs of (not necessarily distinct) elements of V. Throughout out this paper we shall assume that all multi-digraphs have finite vertex and edge sets. Fix a nonnegative integer n. By an *open graph of order* n, we mean a triple $\Gamma = (V, E, \vec{b})$, where (V, E) is a multi-digraph and $\vec{b}$ is an n-tuple of elements of V, called the *boundary vector* of the open graph.

Let W denote a matrix whose rows and columns are indexed by X and whose entries lie in some field $\mathbb{K}$. The elements of the planar algebra generated by W can be expressed by applying the partition function (1) to planar tangles T all of whose internal disks are touched by exactly four strings and are labeled by W. Any such planar tangle T may be turned into an open graph Γ of order k_0. With the notation of subsection 2.3, the vertices of Γ are the labels of the shaded regions of T, the edges are the pairs (S_{i1}, S_{i2}) $(1 \leq i \leq n)$, where S_{ij}, and the boundary vector is $(S_{01}, S_{02}, \ldots S_{0k_0})$. When drawing an open graph, we use points and lines to represent vertices and edges, and we number the boundary vertices with their place in the boundary vector.

Let Γ be an open graph of order k. The partition function of Γ with respect to W is the function $Z_\Gamma^W : X^k \to \mathbb{K}$, which when evaluated at $(x_1, x_2, \ldots x_k) \in X^k$ returns

$$Z_\Gamma^W(x_1, x_2, \ldots, x_k) = \sum_\sigma \prod_{(u,v)\in E} W(\sigma(u), \sigma(v)), \tag{3}$$

where σ runs over all maps from V to X with $\sigma(b_i) = x_i$ $(0 \leq i \leq n)$.

When Γ comes from a planar tangle, (3) agrees with (1). However, (3) makes perfect sense on any open graph. For all i, let $\mathcal{O}_i^W$ denote the vector subspace of $\mathcal{X}_i$ spanned by the functions defined by (3) as Γ runs over all open graphs of order k. We call $\mathcal{O}^W = \cup \mathcal{O}_i^W$ the *open graph planar algebra of* W.

When W is the adjacency matrix of a graph, the partition function has a particularly nice interpretation. Let $\Delta = (V, E)$ and $\Gamma = (X, R)$ be graphs. By a *graph homomorphism* from Δ into Γ, we mean a map $\sigma : V \to X$ such that if $(u, v) \in E$ then $(\sigma(u), \sigma(v)) \in R$.

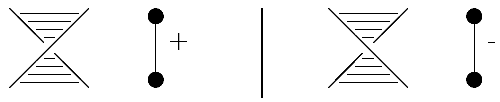
Lemma 1. *Let $\Delta = (X, R)$ denote a digraph, and let W denote the adjacency matrix of Δ. Let $\Gamma = (V, E, \vec{b})$ denote an open graph of order n for some fixed nonnegative integer n. Then for all $\vec{p} \in X^n$, $Z_\Delta^W(\vec{p})$ equals the number of graph homomorphisms from (V, E) to Δ which map $\vec{b}$ to $\vec{p}$ coordinate-wise.*

3. Reidemeister moves as examples of skein relations

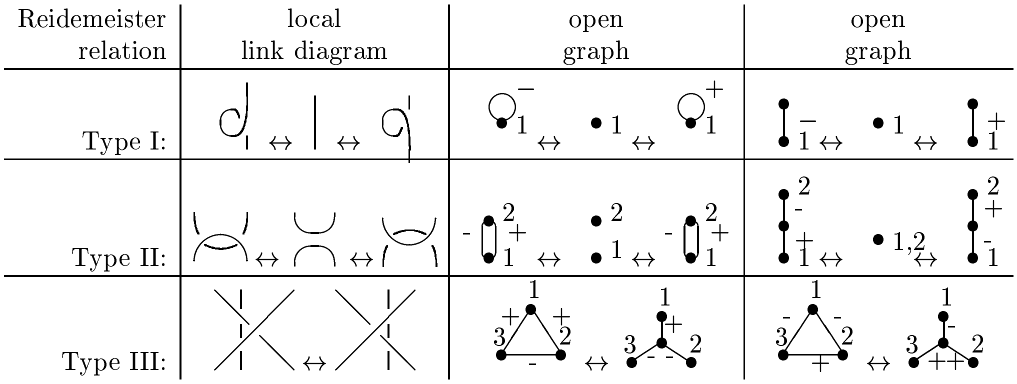
In this section we discuss some examples of the planar algebras $\mathcal{P}^W$, $\mathcal{O}^W$. We emphasize the interpretation of the relations of these planar algebras as graph rewriting rules. Given open graphs Γ_i, we say that $\sum_i \alpha_i \Gamma_i = 0$ modulo W when $\sum_i \alpha_i Z_\Gamma^W = 0$. The relations of planar algebra are similar to the formal combinations of diagrams used by knot theorists, such as Conway's tangles and skein relations.

The prototypical examples are the Reidemeister moves and spin models. We begin with a well-known translation of a link diagram into a planar graph and the corresponding translation of the Reidemeister moves into open graph relations. Given a link diagram, the regions of the plane formed by the diagram are checkerboard shaded black and white, and an undirected graph with edge labels

$\{+,-\}$ is formed where each shaded region is a vertex and each crossing is an edge with sign given by the following rule:

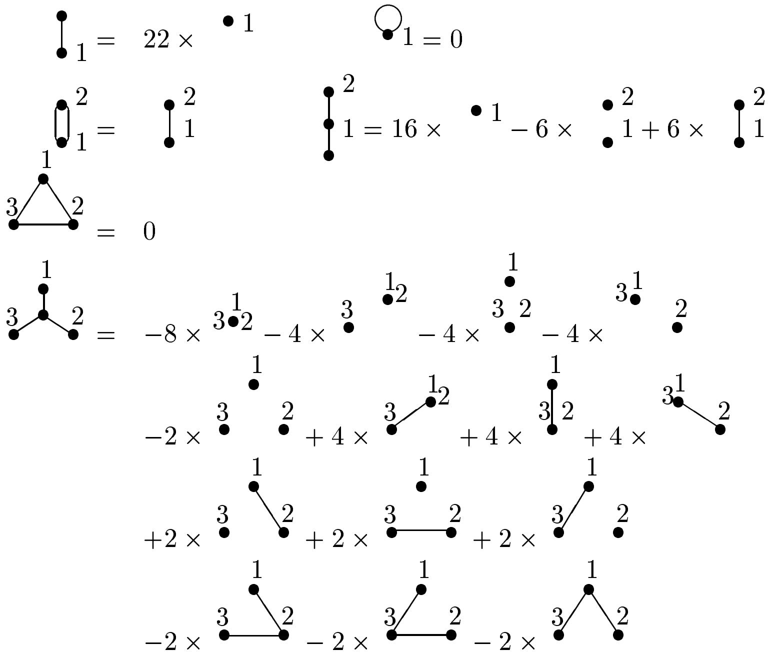


The Reidemeister moves are translated to graph rewriting rules in the following figure. We have omitted the shading of the link diagram. The two possibilities give two open graph relations.



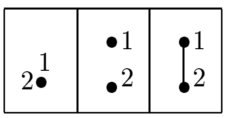
The two columns of open graphs correspond to the two shadings of the link diagram. The boundary vertices are numbered in a clockwise order to remind ourselves that such vertices may have neighbors in the larger graph, while those with no label have no neighbors in the larger graph other than those shown. The numbers also keep track of which vertices replace one another in the application of the particular move. The problem of findings pairs of matrices $\Omega = \{W^+, W^-\}$ such that the planar algebra $\mathcal{P}^\Omega$ generated by these matrices has Reidemeister relations is quite challenging. Such a pair is called a spin model [1, 8, 13].

We now consider singly generated planar algebras with relations like the Reidemeister moves. A most amazing example of a spin model has been constructed by Jaeger using the Higman-Sims graph [7] (see also [6]). Let H denote the adjacency matrix of the Higman-Sims graph. It turns out that $\dim \mathcal{P}^H_1 = \dim \mathcal{F}^H_1 = 1$, $\dim \mathcal{P}^H_2 = \dim \mathcal{F}^H_2 = 3$, $\dim \mathcal{P}^H_3 = \mathcal{F}^H_3 = 14$, but $\dim \mathcal{P}^H_4 = 105$ and $\dim \mathcal{F}^H_4 =$. It follows that $\mathcal{P}^H \neq \mathcal{F}^H$. We prove these formulas for $\dim \mathcal{P}^H_n$ by using some skein relations to show the values are upper bounds, and then produce the appropriate number of open graphs and prove their linear independence. The proof of the general formula for $\dim \mathcal{P}^H_n$ is nontrivial and beyond the scope of the present work. We claim that the following relations hold modulo H:

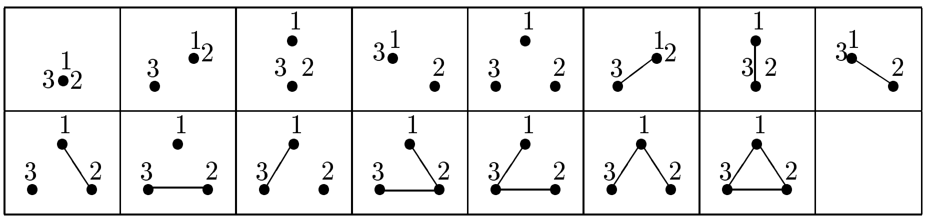


These relations are not exactly the Reidemeister moves (H itself is not a spin model, but rather certain linear combinations of I, J and H). However, they are similar. The first row's relations are type I-like, allowing one to remove loops and pendant edges. The second row's relations are type II-like, allowing one to collapse multiple edges and vertices of degree 2. The last two relations are type III-like, but somewhat stronger because both triangles and stars can be reduced to graphs with fewer edges directly. A modified form of Reidemeister's theorem may be applied. In terms of graphs such a result is a straight forward adaptation of a result of Epifanov [5] to open graphs, and in terms of planar algebras it is implied by a result of [2].

These skein relations allow one to reduce any planar open graph to an "isotopy representative" (a set of planar open graphs corresponding to a set of isotopy representatives of a open link diagram). The number of isotopy representatives of planar open graphs of order n is $(2n-1)(2n-3)\cdots 3\cdot 1$. This is an upper bound on $\dim \mathcal{P}_n^H$. In terms of string diagrams, the isotopy class representatives are described as all ways of joining $2n$ points on a circle with n strings and checkerboard shading. The corresponding open graphs are then produced from these diagrams as above. The isotopy class representatives of open graphs of order 2 are:



The 15 isotopy representatives of open graphs of order 3 are:



The triangle is isotopic with the star, which must be used as the class representative when the triangle is gives the zero function in the planar algebra.

We propose planar algebras as a foundation for studying certain graph rewriting problems. One feature of the skein relations of a planar algebra is that the partition function is an invariant of the system by construction–this fact that allows one to produce a link invariant from a spin model. There are many open problems in this approach to graph rewriting.

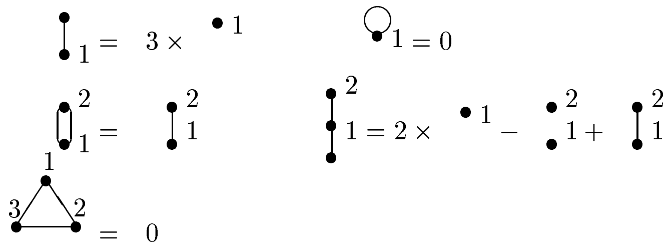
- Find a matrix such that its planar algebra has some given relations.

One may turn the problem around and start with an arbitrary matrix and study its planar algebra. Of course given any matrix W, its planar algebra will have many skein relations because $\mathcal{P}^W$ is finite dimensional while there are infinitely many planar open graphs.

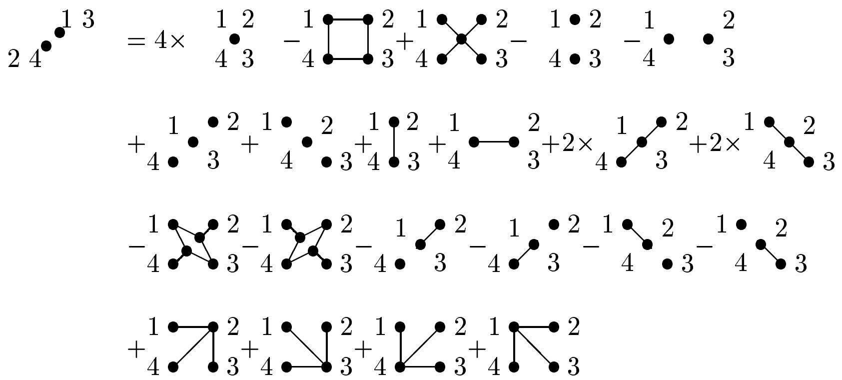
- Find a set B of planar open graphs which define a basis for $\mathcal{P}^W$, a set S of skein relations in $\mathcal{P}^W$, and an algorithm to reduce all planar open graphs to those in B using only the skein relations in S.

The next section discusses a result which is sometimes helpful in determining the dimensions of the levels in a planar algebra, and thus deciding if one has a basis B or not.

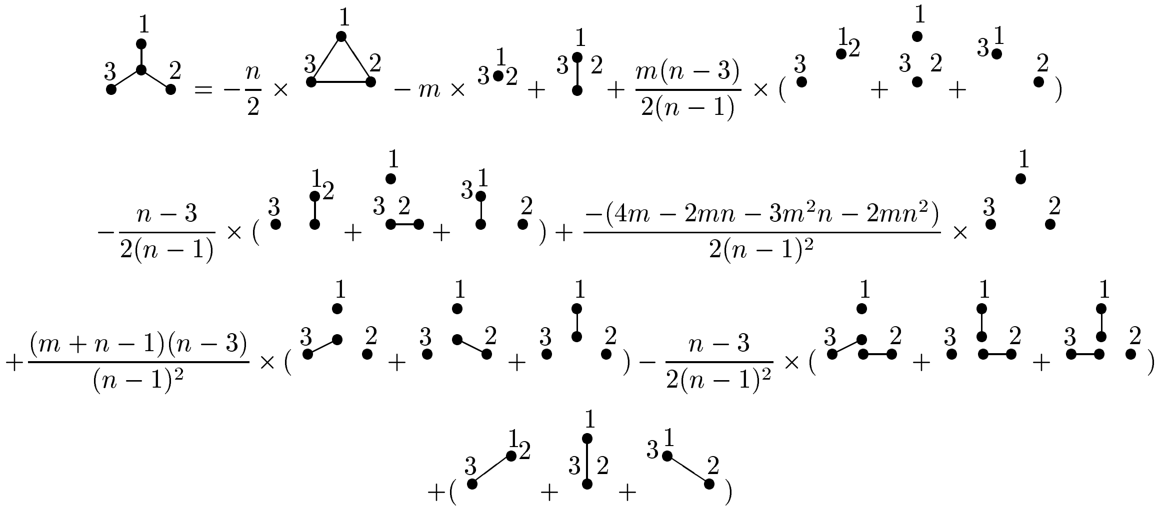
For the sake of comparison, we consider the Petersen graph. Let P denote the adjacency matrix of the Petersen graph. Then modulo P the following relations hold:



(In fact, we have similar relations for any triangle-free strongly regular graph–the Higman-Sims and Petersen graphs are examples). This is quite similar to the Higman-Sims graph. In fact, we also have $\dim \mathcal{P}_1^P = \dim \mathcal{F}_1^P = 1$, $\dim \mathcal{P}_2^P = \dim \mathcal{F}_2^P = 3$, $\dim \mathcal{P}_3^P = \dim \mathcal{F}_3^P = 15$. It is interesting to note that $\dim \mathcal{F}_4^P = 107$, just two more than the isotopy bound of 105. It turns out that $\dim \mathcal{P}_4^P = 107$. This can be shown by simply producing 107 planar open graphs which define linearly independent functions. In fact, $\mathcal{P}^P = \mathcal{F}^P$. But this requires a much more sophisticated approach. In the next section we shall discuss a means of checking this condition by showing that the function defined by a certain nonplanar open graph of belongs to $\mathcal{P}_4^P$. This is verified as follows:



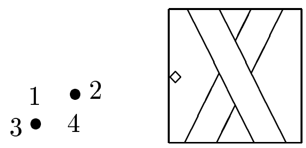
Having had some success with Reidemeister-like relations, one may wish to find other examples. The Petersen graph is a near miss. We observe that the type I and type II-like relations hold whenever one starts with the adjacency matrix of a strongly regular graph. A type III-like relation in the planar algebra of a strongly regular graph occurs when the graph is "triply regular", a very rare condition indeed. The Higman-Sims graph has this property. It is not known if there are any further examples. The planar algebra generated by one of the matrices in a spin model is quite similar, but unless the spin model is related to a strongly regular graph the planar algebra is not as simple as the Higman-Sims case. In the singly generated planar algebras, a modification of Epifanov's result [5] shows that the type III-like relations which plays the most important role in reductions. By giving up the type I and II-like relations we may find other graphs whose planar algebras have type III-like relations. For example, let W be the adjacency matrix of the graph consisting of a clique of size $m \geq 3$ and a coclique of size $n \geq 3$ with all possible edges between these two sets. Then modulo W pendant edges cannot be reduced, but there is a type III-like relation:



4. When $\mathcal{P}^W = \mathcal{F}^W$

Observe that $\mathcal{P}^W \subseteq \mathcal{O}^W \subseteq \mathcal{F}^W$. We recall from [4] results describing when $\mathcal{P}^W = \mathcal{O}^W$ and when $\mathcal{O}^W = \mathcal{F}^W$. (They are also related to a result of [11]). Together they give the test of when $\dim \mathcal{P}_n^W$ is given by the group bound (2) which we used above on the Petersen graph.

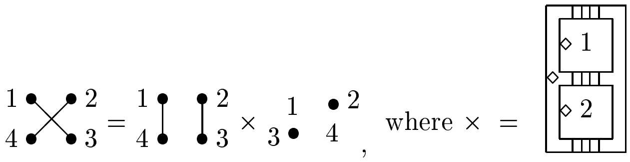
Of course $\mathcal{P}^W = \mathcal{O}^W$ when the partition function of every open graph is in span of the partition functions of the planar tangles. In fact, the problem reduces to showing that this is the case for one very simple (non-planar) open graph. Let Φ denote the (non-planar) open graph of order 4 consisting two isolated vertices v_1 and v_2 and boundary vector (v_1, v_2, v_1, v_2). We call Φ the *transposition*. We can draw Φ as a (non-planar) 4-tangle by drawing two "ribbons" which cross, one above the other, without interacting.



(a) open graph (b) tangle

Theorem 1. *Pick $W \in \mathcal{M}_{\mathbb{K}}(X)$ Then $\mathcal{P}^W = \mathcal{O}^W$ if and only if $Z_\Phi^W \in \mathcal{P}_4^W$.*

The forward implication is clear. The proof of the reverse implication relies on pictorial manipulation. First, by repeatedly transposing adjacent boundary vectors, it is possible to rewrite any open graph as a linear combination of an open graphs with the property that any repetitions in the boundary vector appear in cyclically successive positions. Thus we only need to show that the partition function of such an open graph Γ is in the span of partition functions of open graphs coming from planar tangles. Embed Γ into the plane so that its vertices lie on a circle respecting the order of the boundary vertices. Then crossing edges are factored into noncrossing edges and a transposition. This procedure is done by rewriting graphs as linear combinations of other graphs with the reordered vertices and crossings removed. Finally, the transposition itself is replaced by a linear combination of planar open graphs, as allowed by the assumption that $Z_\Phi^W \in \mathcal{P}_4^W$.



Theorem 2. *Let $\mathbb{K}$ denote a field, let X denote a finite nonempty set, and pick $W \in \mathcal{M}_{\mathbb{K}}(X)$. Suppose that $|X|!$ is not divisible by the characteristic of $\mathbb{K}$. Then $\mathcal{O}^W = \mathcal{F}^W$.*

The basic idea of the proof is to first fix $\vec{p} \in X^n$, and to study the set of polynomials $Z_\Gamma^L(\vec{p}) - Z_\Gamma^W(\vec{p})$ as Γ runs over all open graphs of order n, where L is the matrix whose (u, v)-entry is a variable ℓ_{uv} $(u, v \in X)$. It turns out that the $L = W$ is essentially the unique simultaneous zero of these polynomials, up to a trivial group action which fixes each entry of $\vec{p}$ (this uses the assumption on the field characteristic). Now if $\vec{q} \in X^n$ is such that $Z_\Gamma^W(\vec{p}) = Z_\Gamma^W(\vec{q})$ for all open graphs Γ of order n, then there is a permutation σ of X such that $\vec{q} = \sigma(\vec{p})$. Moreover, σ must factor as a permutation of X which fixes each entry of $\vec{p}$ followed by a permutation which leaves W invariant when it simultaneously permutes the rows and columns of W. In particular, $\vec{p}$ and $\vec{q}$ belong to the same $\mathfrak{Aut}(W)$-orbit of X^n. It follows that $\mathcal{O}^W = \mathcal{F}^W$.

In general, we work in characteristic zero where the condition of Theorem 2 is always satisfied. In positive characteristic some condition on the field is necessary. When W is the adjacency matrix of a complete multipartite graph $K_{n_1, n_2, \ldots, n_m}$ over a field $\mathbb{K}$ of characteristic $k > 0$, $\dim \mathcal{O}_1^W$ is equal to the number of congruence classes modulo k appearing among $n_1, n_2, \ldots, n_m$ while $\dim \mathcal{F}_1^W$ is equal to the number of distinct numbers among $n_1, n_2, \ldots, n_m$.

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