LATTICE PATH MOMENTS BY CUT AND PASTE

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ABSTRACT. In the coordinate plane consider those lattice paths whose step types consist of $(1,1)$, $(1,-1)$, and perhaps one or more horizontal steps. For the set of such paths running from $(0,0)$ to $(n+2,0)$ and remaining strictly elevated above the horizontal axis elsewhere, we define a zeroth moment (cardinality), a first moment (essentially, the total area), and a second moment in terms of the ordinates of the lattice points traced by its paths. We then establish a bijection relating these moments to the cardinalities of sets of selected lattice points on the unrestricted paths running from $(0,0)$ to $(n,0)$. Roughly, this bijection acts by *cutting* each elevated path into well-defined subpaths and then *pasting* the subpaths together in a specified order to form an unrestricted path.

RÉSUMÉ. Nous considérons les chemins du plan dont les pas sont (1,1), (1,-1) et éventuellement un ou plusieurs pas horizontaux. Pour l'ensemble des chemins de ce type, qui restent au dessus de l'axe des abscisses et qui vont de (0,0) à (n+2,0) nous définissons des moments d'ordre 0 (cardinalité), d'ordre 1 (aire) et d'ordre 2, en utilisant les ordonnées des points qui appartiennent au chemin. Nous établissons une bijection entre ces moments et les cardinalités de certains ensembles de points sur les chemins sans restriction allant de (0,0) à (n,0). Cette bijection fonctionne en coupant les chemins primitifs en sous chemins et en les recollant dans un certain ordre pour former le chemin sans restriction associé.

1. THE PATHS AND THEIR MOMENTS

In the plane of lattice points, $\mathbb{Z}\times\mathbb{Z}$, consider a lattice path to be sequences of steps. Let $U := (1,1)$, the up-diagonal step, let $D := (1,-1)$, the down-diagonal step, and let $\mathcal{H}$ denote a fixed set of steps of integral lengths, i.e., $\mathcal{H} := \{(h_1,0),(h_2,0),\ldots\}$. Given a set of permitted step types, $\{U,D\}\cup\mathcal{H}$, let $\mathcal{U}(n)$ denote the set of all *unrestricted* paths running from $(0,0)$ to $(n,0)$. Let $\mathcal{E}(n)$ denote the subset of those paths in $\mathcal{U}(n)$ that are *elevated* strictly above the x-axis except initially and finally. Fig. 1 illustrates the set of all paths in $\mathcal{E}(6)$ for $\mathcal{H} = \{(2,0)\}$.

When we allow only the steps U and D, i.e., when $\mathcal{H} = \emptyset$, the path set $\mathcal{E}(2n+2)$ is the set of the so-called *elevated Dyck* paths running from $(0,0)$ to $(2n+2,0)$. The respective cardinalities of $\mathcal{E}(2n+2)$ and $\mathcal{U}(2n)$ are the Catalan numbers and central binomial coefficients, which are related by

$$|\mathcal{E}(2n+2)| = \frac{1}{n+1}|\mathcal{U}(2n)| = \frac{1}{n+1}\binom{2n}{n}. \tag{1}$$

The cut and paste bijection will explain the ratio $1/(n+1)$, but not quite as the classical cycle lemma did [1, 2].

For a general set of horizontal steps $\mathcal{H}$ and for any path P, let the points

$$(0,p_0),(1,p_1),(2,p_2),\ldots,(x,p_x),\ldots,(n,p_n)$$

be the lattice points (not just end points of steps) traced by the path. We introduce three moments: For $j = 0, 1$, and 2, consider the sum of the j^{th} moments of the paths of $\mathcal{E}(n)$ defined by

$$\mu(n,j) := \sum_{P\in\mathcal{E}(n)} \frac{1}{n-1}\sum_{0<i<n}(p_i)^j.$$

Immediately we have $|\mathcal{E}(n)| = \mu(n,0)$. By the elementary trapezoid rule we have that $(n-1)\mu(n,1)$ is the total area of the regions bounded by the paths of $\mathcal{E}(n)$ and the x-axis. For an example, in the

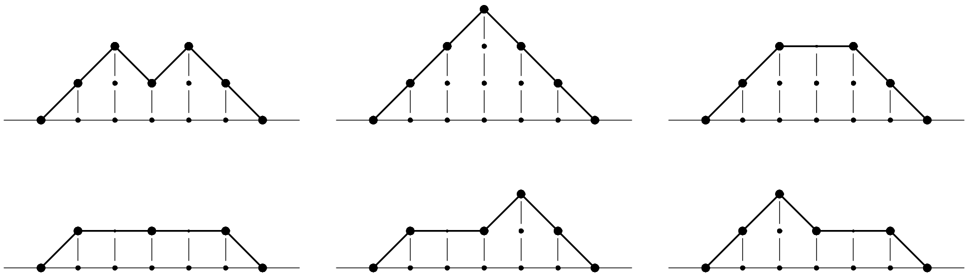

FIGURE 1. The six Schroder paths of $\mathcal{E}(6)$ for $\mathcal{H} = \{(2,0)\}$.

case for the elevated Schröder paths in $\mathcal{E}(6)$, i.e, paths where the U, D, and $(2,0)$ are the permitted steps, one can calculate from Fig. 1 that $\mu(6,0) = 6$, $\mu(6,1) = 41/5$ with a total area of 41, and $\mu(6,2) = 13$.

In the case of the elevated Dyck path, i.e. when $\mathcal{H} = \emptyset$, we have

$$\mu(2n+2,0) = \frac{1}{n+1}\binom{2n}{n} \tag{2}$$

$$\mu(2n+2,1) = \frac{4^n}{2n+1} \tag{3}$$

$$\mu(2n+2,2) = \binom{2n}{n} \tag{4}$$

Formula (2) is equivalent to (1) and is re-proven in Proposition 4. Formula (3) is well-known (See [3] and [8].) and will be considered in Corollary 3.

In (4) we have the remarkable result that the sum of the average height squared is a central binomial coefficient. We conjecture that, for any set of horizontal steps $\mathcal{H}$, the second moment is the cardinality of an unrestricted path set; specifically,

$$\sum_{P\in\mathcal{E}(n+2)} \frac{1}{n+1} \sum_{0<i<n+2} (p_i)^2 = |\mathcal{U}(n)|. \tag{5}$$

In [6] generating function methods yield this result for paths using the step set, $\{U,\ D,\ (h,0)\ \}$, for any fixed positive integer h. In [5] and [7], bijective methods explain this result for Motzkin paths with the steps in $\{U, D, (1,0)\}$ and Schröder paths with the steps in $\{U, D, (2,0)\}$, respectively. In each of these three papers, both sides of (5) are shown to satisfy a recurrence of the same form, and, hence by induction, the sides are equal. A main purpose of this paper is to give a direct bijective proof of (5) without using induction.

In the next section, we will translate the moments into configurations of indexed lattice points lying under the paths of $\mathcal{E}(n+2)$. We will refer to each such indexed lattice point as a *dot*. In Section 3, extending a method of [3], we introduce our *cut and paste* bijection which will relate the moments of elevated paths, represented by sets of *dots*, to the cardinalities of sets of specific lattice points lying on unrestricted paths. Roughly, this bijection maps *dots* under the elevated paths to lattice points on unrestricted paths. It does so by *cutting* each elevated path into two or four well-defined subpaths and then *pasting* these subpaths together in a specific order to create an unrestricted path. In Section 4 we will use selected sets of *dots* to produce various results relating the three moments to unrestricted path sets. The surprising result here is the bijective proof of Proposition 1 relating the average height squared of elevated paths to the cardinalities of unrestricted paths. Also in that section we will see how the *cut and paste* bijection relates to the cycle lemma of Dvoretzky and Motzkin [2].

2. The dots

Here we introduce the notion of a *dot lying under a path* of $\mathcal{E}(n+2)$. Given an elevated path P, given a lattice point (x, y) lying strictly under the path P and weakly above the x-axis, and given an integer parameter k, a *dot* is just a copy of a lattice point (x, y) indexed by k. We will denote such a dot by $[P,(x,y),k]$. Since the dots are indexed, we can distinguish between dots having the same location.

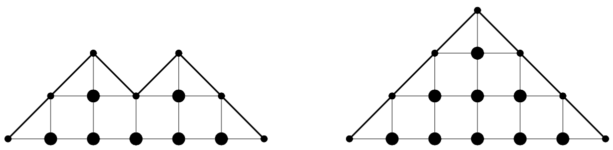

FIGURE 2. The 4^2 dots under the elevated Dyck paths of $\mathcal{E}(6)$.

When there is exactly one dot at each point lying strictly under a given path, the trapezoid rule implies that the number of dots is equal to the area of the region bounded by the path and the x-axis. In terms of dots, the result in (3) for the total area under the elevated Dyck paths becomes, with $k = 0$,

$$|\{[P,(x,y),0] : P \in \mathcal{E}(2n+2), 0 < x < n+2, 0 \leq y < p_x\}| = 4^n.$$

In Fig. 2 the sequences $(0,0)$,$(1,1)$,$(2,2)$,$(3,1)$,$(4,2)$,$(5,1)$,$(6,0)$ and $(0,0)$,$(1,1)$, (2,2), (3,3), (4,2), (5,1), (6,0) represent the lattice points traced by the two Dyck paths of $\mathcal{E}(6)$. The collection of the 4^2 dots corresponds to the total area under these paths.

Now consider the second moment result in (5), or equivalently

$$\sum_{P\in\mathcal{E}(n+2)} \sum_{0<i<n+2} (p_i)^2 = (n+1)|\mathcal{U}(n)|. \tag{6}$$

For orientation to our main bijection, which appears in the next section and which almost immediately yields the result of (6), we now introduce the objects constituting its domain and codomain. The left side of identity (6) will correspond to a sequence of square arrays of dots such that beneath each lattice point (x, p_x) on each path P of $\mathcal{E}(n+2)$ there is a p_x by p_x array of dots. The product of the right side of (6) will correspond to the collection of all lattice points traced by all paths of $\mathcal{U}(n)$. Fig. 3 illustrates the case for $\mathcal{E}(6)$ and $\mathcal{U}(4)$ when $\mathcal{H} = \emptyset$. Here $\mathcal{U}(2n) = \{UUDD, UDUD, UDDU, DUUD, DUDU, DDUU\}$. The top two paths illustrate the 30 dots for $\mathcal{E}(6)$ corresponding to the left side of (6) while the bottom six paths show the 30 lattice points on the paths of $\mathcal{U}(4)$.

Let us consider further the square array of dots lying under an arbitrary lattice point (x, p_x). Any square array of dots can be transformed to a triangular array of dots. For example, if $(x, p_x) = (x, 3)$ is a point of ordinates 3 on a path P, then such a transformation is illustrated in Fig. 4.

For a given path P and abscissa x we formalize this transformation by defining

$$\Delta : \{[P,(x,y),k] : 0 \leq y < p_x, 0 \leq k < p_x\} \to \{[P,(x,y),k'] : -p_x < k' < p_x, 0 \leq y < p_x - |k'|\}$$

where

$$\Delta([P,(x,y),k]) = \begin{cases} [P,(x,y),k] & \text{if } \ 0 \leq y < p_x - k \\ [P,(x,y-p_x+k), -p_x+k] & \text{if } \ p_x - k \leq y < p_x \end{cases}$$

For example, with $P = UUUDDD$ and $x = 3$ and with $[x, y, k]$ abbreviating $[P,(x,y),k]$, this transformation appears as

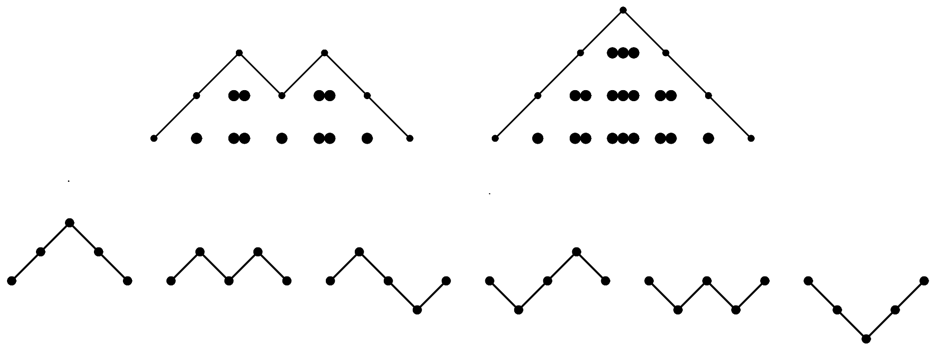

FIGURE 3. The 30 *dots* representing the second moment of the two elevated paths correspond to the 30 *points* on the unrestricted paths.

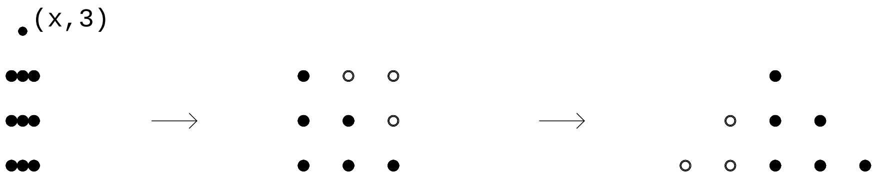

FIGURE 4. Transforming a 3 by 3 square of dots into a triangle of dots.

$$\begin{array}{ccc} [3,2,0] & [3,2,1] & [3,2,2] \\ [3,1,0] & [3,1,1] & [3,1,2] \\ [3,0,0] & [3,0,1] & [3,0,2] \end{array} \quad \longrightarrow \quad \begin{array}{ccccc} & & [3,2,0] & & \\ & [3,1,-1] & [3,1,0] & [3,1,1] & \\ [3,0,-2] & [3,0,-1] & [3,0,0] & [3,0,1] & [3,0,2] \end{array}$$

3. THE CUT AND PASTE BIJECTION

We now define the cut and paste bijection, denoted by

$$\phi : \mathrm{DOTS}(n+2) \to \mathrm{POINTS}(n), \tag{7}$$

for $n \geq 0$. Here the domain is the collection of dots:

$$\mathrm{DOTS}(n+2) := \{[P,(x,y),k] : P \in \mathcal{E}(n+2), 0 < x < n, -p_x < k < p_x, 0 \leq y < p_x - |k|\}.$$

As in the previous section, this collection consists of triangular arrays of dots, one for each lattice point on each elevated path. The codomain is the collection of all lattice points traced by the paths of $\mathcal{U}(n)$:

$$\mathrm{POINTS}(n) := \{[P,(x,p_x)] : P \in \mathcal{U}(n), 0 \leq x \leq n\},$$

which is essentially $\mathcal{U}(n) \times \{0,1,\ldots,n\}$.

First assume $k \geq 0$. Given any dot, $[P,(x,y),k] \in \mathrm{DOTS}(n+2)$, we can determine four points on P:

– Let θ be the point on P directly above (x,y); i.e., $\theta = (x,p_x)$.

– Let $\epsilon = (\epsilon_1,\epsilon_2)$ be the nearest point on P to the left of (x,y) having ordinate equal to $p_x - k - 1$. Notice the important role that k plays in defining the bijection.

– Let $\lambda = (\lambda_1,\lambda_2)$ be the nearest point on P to the left of (x,y) having ordinate equal to y.

– Let $\rho = (\rho_1,\rho_2)$ be the nearest point on P to right of (x,y) having ordinate equal to y.

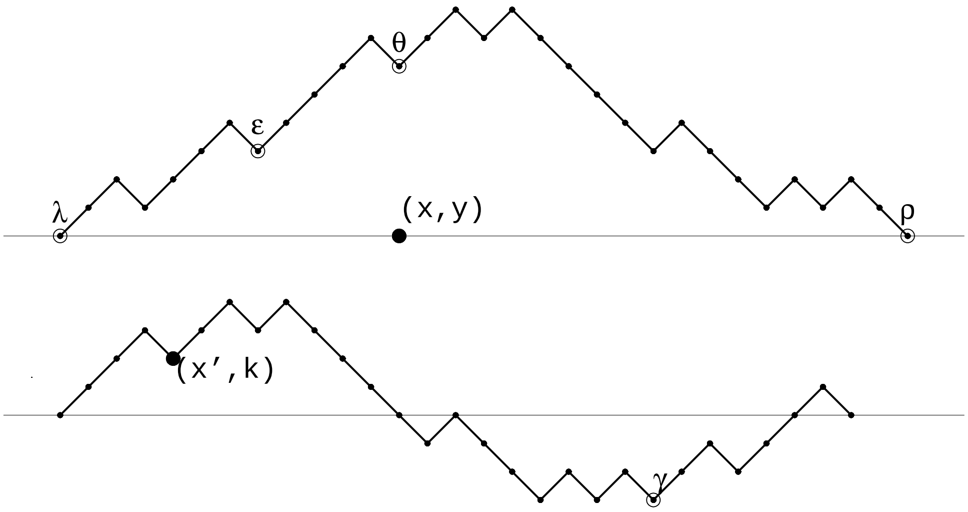

FIGURE 5. For **Case 1**: With $k = 2$, the dot $[P,(12,0),2]$ in DOTS(30) and its image $[P',(x',k)]$ in POINTS(28).

Case 1: Suppose $y = 0$. Immediately $\lambda = (0,0)$ and $\rho = (n+2,0)$. See the top path in Fig. 5 where $k = 2$. Let L_2 be that subpath of P running from λ to ϵ and let R_1 be that subpath of P running from ϵ to ρ. Here $P = L_2\ R_1$. Let $\overline{\overline{R_1}}$ be that path obtained from R_1 by deleting its first and last steps. Define

$$\phi([L_2\ R_1,(x,y),k]) = [\overline{\overline{R_1}}\ L_2,(x',k)] \tag{8}$$

where the point θ is moved along with $\overline{\overline{R_1}}$ so that $x' = x - \epsilon_1 - 1$. For this case we **observe** that, on the left of (x',k), no part of the concatenation $P' = \overline{\overline{R_1}}\ L_2$ lies below the x-axis.

Case 2: Suppose $y > 0$, as in the top path of Fig. 6. Immediately $0 < \lambda_1$ and $\rho_1 < n+2$. Let L_1 be that subpath of P running from $(0,0)$ to λ. Let L_2 be that subpath of P running from λ to ϵ. Let R_1 be that subpath of P running from ϵ to ρ. Let R_2 be that subpath of P running from ρ to $(n+2,0)$. Here $P = L_1\ L_2\ R_1\ R_2$. Let $\overline{L_1}$ be that path obtained from L_1 by deleting its first step. Let $\overline{R_1}$ be that path obtained from R_1 by deleting its last step. Define

$$\phi([L_1\ L_2\ R_1\ R_2,(x,y),k]) = [R_2\ \overline{L_1}\ \overline{R_1}\ L_2,(x',k)] \tag{9}$$

where the point θ is moved along with R_1 so that $x' = x + n + \lambda_1 - \epsilon_1 - \rho_1 - 1$. The factorization $L_1\ L_2\ R_1\ R_2$ is the *cut* and $P'R_2\ \overline{L_1}\ \overline{R_1}\ L_2$ results from the *pasting*. **Observe** that part of P' now lies below the x-axis to the left of (x',k).

If $k < 0$, let REFL(P') denote the reflection of the path P' about the x-axis and define

$$\phi([P,(x,y),k]) = [\mathrm{REFL}(P'),(x',k)]$$

where P' is the path defined by $\phi([P,(x,y),|k|]) = [P',(x',|k|)]$.

To see that the map ϕ is indeed bijective it is sufficient to consider a candidate for its inverse:

$$\psi : \mathrm{POINTS}(n) \to \mathrm{DOTS}(n+2).$$

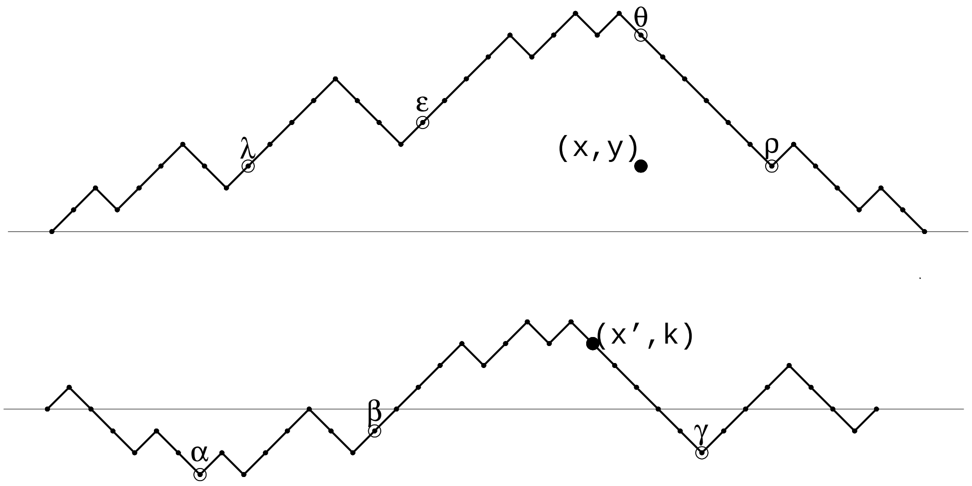

FIGURE 6. For **Case 2**: With $k = 3$, the dot $[P, (27, 3), 3]$ in DOTS(40) and its image $[P', (25, 3)]$ in POINTS(38).

For $k \geq 0$ we define ψ in two **Cases**, each corresponding to the final **observation** in each **Case** in the definition for ϕ. The case for $k < 0$ is handled by reflection as before.

Given $[P', (x', k)] \in \text{POINTS}(n)$, we can determine two points on P':

– Let $\alpha = (\alpha_1, \alpha_2)$ be the point on P' that is the left-most of the lowest points to the left of (x', k).

– Let $\gamma = (\gamma_1, \gamma_2)$ be the point on P' that is the right-most of the lowest points to the right of (x', k).

Case I: Suppose $\alpha_2 = 0$. Let C_1 be the subpath of P' running from $(0, 0)$ to γ, and let C_2 be the subpath of P' running from γ to $(n, 0)$. Define

$$\psi([C_1C_2, (x', k)]) = [UC_2C_1D, (x' + n + 1 - \gamma_1, k + 1 - \gamma_2), k].$$

Case II: Suppose $\alpha_2 < 0$. Let β be the nearest point to the left of (x', k) having ordinate equal to -1. Let A_1 be the subpath of P' running from $(0, 0)$ to α. Let A_2 be the subpath of P' running from α to β. Let C_1 be the subpath of P' running from β to γ. Let C_2 be the subpath of P' running from γ to $(n, 0)$. Then define

$$\psi([A_1A_2C_1C_2, (x', k)]) = [UA_2C_2C_1DA_1, (x' + n + 1 - \gamma_1, k + 1 - \gamma_2), k].$$

4. Applying cut and paste to selected sets of dots

Here we will use the cut and paste method to prove known results. First, without restricting the domain of the cut and paste bijection, we will establish a result for the second moments of elevated paths. Next, by restricting its domain we will obtain results for the first and then the zeroth moments.

Keeping the full set, DOTS$(n + 2)$, for the domain of ϕ, the transformation Δ of Section 2 immediately yields a bijective proof of (6), repeated as

Proposition 1. *For any step set $\{U, D\} \cup \mathcal{H}$, and for $n \geq 0$,*

$$\sum_{P \in \mathcal{E}(n+2)} \sum_{i=1}^{n+1} (p_i)^2 = (n + 1)|\mathcal{U}(n)|.$$

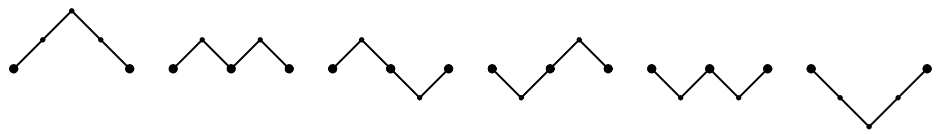

FIGURE 7. The set of 4^2 points in POINTS(4) corresponding to the dots of Fig. 2.

By restricting our bijection to

$$\phi : \{[P,(x,y),0] : P \in \mathcal{E}(n+2), 0 < x < n+2, 0 \le y < p_x\}$$
$$\to \{[P',(x',0)] : P' \in \mathcal{U}(n), 0 \le x' \le 2n\}$$

we have a bijective proof that the total area under the paths of $\mathcal{E}(n+2)$ equals the cardinality of the intercepts of the x-axis by the paths of $\mathcal{U}(n)$. Here the parameter $k = 0$. See Fig. 7 with Fig. 2.. More generally, we will prove

Proposition 2. *For $k \ge 0$, $n \ge 0$, and steps set $\{U,D\} \cup \mathcal{H}$, the total area of the regions under the paths of $\mathcal{E}(n+2)$ and above the line $y = k$ is equal to the cardinality of the points on the paths of $\mathcal{U}(n)$ having ordinate k.*

To prove this, observe that the regions being considered correspond to having exactly one dot at each lattice point lying weakly above the line $y = k$. For each path P and abscissa x, either $p_x > k$ and there is a column of $p_x - k$ such dots, or $p_x \le k$ and there are no dots. Pushing the column downward in a rigid fashion until its lowest dot hits the x-axis yields the following column of dots, where we assign the parameter k:

$$[P,(x,0),k],\ [P,(x,1),k], \dots, [P,(x,p_x-k-1),k].$$

Now, with k fixed, the proof is completed by restricting ϕ to

$$\phi : \{[P,(x,y),k] : P \in \mathcal{E}(n+2), 0 < x < n+2, 0 \le y < p_x - k\}$$
$$\to \{[P',(x',k)] : P' \in \mathcal{U}(2n), 0 \le x' \le 2n\}.$$

Corollary 3. *For the step set $\{U,D\}$, the total area of the regions under the paths of $\mathcal{E}(n+2)$ is 4^n.*

Our proof for this was suggested by Louis Shapiro and appears with more detail in [3]. Consider the generating function, $\sum_{n\ge0} K_n z^n$, where K_n denotes the sum, over all paths $P \in \mathcal{U}(n)$, of the number of times P contains a point with ordinate 0. Such a point is the final point of one unrestricted path and is the beginning of a second unrestricted path. Hence,

$$\sum_{n\ge0} K_n z^n = \sum_{n\ge0}\sum_{0\le k\le n} |\mathcal{U}(k)||\mathcal{U}(n-k)|z^n = \left(\sum_{n\ge0}|\mathcal{U}(n)|z^n\right)^2 = \left(\frac{1}{\sqrt{1-4z^2}}\right)^2 = \frac{1}{1-4z^2},$$

since the coefficients of $1/\sqrt{1-4z^2}$ are central binomial coefficients which count unrestricted paths.

The next restriction shows that our bijection yields the result of the classical cycle lemma of Dvoretzky and Motzkin [2] and, hence, yields a formula for the Catalan numbers as in Corollary 5 below.

Proposition 4. *For $m \ge 1$, $n \ge 0$, and any step set $\{U,D\} \cup \mathcal{H}$, let $\mathcal{E}(n,m)$ and $\mathcal{U}(n,m)$ denote subset of $\mathcal{E}(n)$ and $\mathcal{U}(n)$, respectively, where each path has m up-diagonal steps. Then*

$$(m+1)|\mathcal{E}(n+2,m+1)| = |\mathcal{U}(n,m)|$$

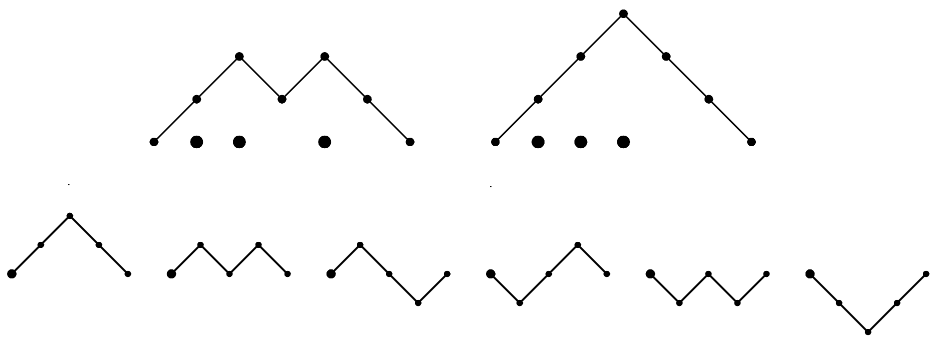

FIGURE 8. The cut and pasting producing Corollary 5.

To prove Proposition 4, place $m+1$ dots on the x-axis under each path P in $\mathcal{E}(n+2, m+1)$ so one dot is located directly below the final point of each up-diagonal step. (See Fig. 8.) With $k=0$ and $y=0$, each dot is mapped by ϕ under Case 1 to a point $\phi([P,(x,0),0]) = [\overline{\overline{R_1}}L_2,(0,0)]$. Considering the restriction

$$\begin{aligned}\phi : \{[P,(x,0),0] : P \in \mathcal{E}(n+2, m+1), x \text{ below a final pt. of a } U\} \\ \to \{[P',(0,0)] : P' \in \mathcal{E}(n,m)\},\end{aligned}$$

we see that the cardinality of the domain is $(m+1)|\mathcal{E}(n+2,m+1)|$ while the cardinality of the codomain is $|\mathcal{U}(n,m)|$.

Corollary 5. *If only the step types U and D are permitted, the Catalan numbers appear as*

$$|\mathcal{E}(2n+2)| = \frac{1}{(n+1)}|\mathcal{U}(2n)| = \frac{1}{(n+1)}\binom{2n}{n}.$$

Corollary 6. *If only the step types U, D, and $(1,0)$ are permitted, the Motzkin numbers appear as*

$$|\mathcal{E}(n+2)| = \sum_{m\geq 0} |\mathcal{E}(n+2,m+1)|$$

$$= \sum_{m\geq 0} \frac{1}{(m+1)}|\mathcal{U}(n,m)| = \sum_{m\geq 0} \frac{n!}{m!\;(m+1)!\;(n-2m)!}.$$

We conclude by noting how our proof of Proposition 4 differs from a common, special form of the cycle lemma, which we state in lattice path notation as follows: *Let P be any unrestricted path running from $(0,0)$ to $(n+1,1)$, consisting of $m+1$ U steps and m D steps, and beginning with a U step. Let $\langle P\rangle$ denote the set of all cyclic permutations of P which begin with a U step. Then there is exactly one path Q in $\langle P\rangle$ which is strictly above the x-axis except initially.*

In essence, proofs of this lemma often *start with an arbitrary path UA*, with $A \in \mathcal{U}(n)$, and then derive a unique path UB in $\langle UA\rangle$ for which UBD is an elevated path in $\mathcal{E}(n+2)$. (The cycle lemma was introduced most prominently in [2] and in [4]. See [1] for an interesting survey; [7] contains a proof of the cycle lemma for Motzkin paths.)

Our proof of Proposition 4, on the other hand, *starts with an elevated path $UBD \in \mathcal{E}(n+2)$* and then applies the cut and paste to that path $m+1$ times producing the paths of the form

$$\phi([UBD,(x_1,0),0]),\ \phi([UBD,(x_2,0),0]),\ \ldots,\ \phi([UBD,(x_{m+1},0),0])$$

in $\mathcal{U}(n)$. Moreover, as is easily checked, the prepended paths

$$U\phi([UBD,(x_1,0),0]),\ U\phi([UBD,(x_2,0),0]),\ \ldots,\ U\phi([UBD,(x_{m+1},0),0])$$

form the cyclic permutation class $\langle UB\rangle$ of paths running from $(0,0)$ to $(n+1,1)$.

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