ON NON P-RECURSIVENESS OF NUMBERS OF MATCHINGS (LINEAR CHORD DIAGRAMS) WITH MANY CROSSINGS

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ABSTRACT. The number con_n counts matchings X on $\{1,2,\ldots,2n\}$, which are partitions into n two-element blocks, such that the crossing graph of X is connected. Similarly, cro_n counts matchings whose crossing graph has no isolated vertex. (If it has no edge, Catalan numbers arise.) We prove, using a more generally aplicable criterion, that the sequences (con_n) and (cro_n) are not P-recursive. On the other hand, we show that the residues of con_n and cro_n modulo any fixed power of 2 can be determined P-recursively. We consider also numbers sco_n of symmetric connected matchings. Unfortunately, their OGF satisfies a more complicated differential equation which we cannot handle.

RÉSUMÉ. Un couplage d'un ensemble $\{1,2,\ldots,2n\}$ est une partition de cet ensemble en sous-ensembles de deux éléments. Soit con_n le nombre de couplages de $\{1,2,\ldots,2n\}$ dont le "graphe des croisements" est connexe. De faon similaire, on note cro_n le nombre de couplages dont le graphe des croisements ne possde pas de sommet isolé. (Les couplages dont le graphe des croisements n'a pas d'arte sont comptés par les nombres de Catalan.) Nous montrons, en applicant un critre généralisable d'autres problèmes, que les suites (con_n) et (cro_n) ne sont pas P-récursives. En revanche, nous montrons que les résidus de con_n et de cro_n modulo n'importe quelle puissance de 2 fixée peuvent tre déterminés P-récursivement. Nous considérons aussi le nombre sco_n de couplages symétriques connexes. La série génératrice ordinaire de ces couplages satisfait une équation différentielle plus compliquée, que nous n'avons pas pu résoudre.

Matchings on the vertex set $[2n] = \{1,2,\ldots,2n\}$ consist of n mutually disjoint two-element edges. One easily finds that their number mat_n equals $(2n-1)!! = 1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)$. Another classical result tells us that the number ncr_n of *noncrossing matchings* on $[2n]$ (no two edges $\{a,b\}$ and $\{c,d\}$ satisfy $a<c<b<d$) is the nth Catalan number: $\mathrm{ncr}_n = \frac{1}{n+1}\binom{2n}{n}$. How many matchings are there if their crossings are restricted in a more complicated way? In the present article we investigate numbers of such matchings, namely the numbers con_n of *connected matchings* in which each two edges can be connected by a chain of consecutivly crossing edges, the numbers sco_n of *symmetric connected matchings* which, in addition, are mirror symmetric, and the numbers cro_n of *crossing matchings* in which each edge crosses another edge. We concentrate only on P-recursiveness of these numbers. Also, we touch upon some congruence properties. The sequences (mat_n) and (ncr_n) are trivially P-recursive but, as we prove, the sequences (con_n) and (cro_n) are not.

First we remind the definition of P-recursiveness and D-finiteness. Then we introduce $\mathrm{D_A}$-finiteness and review some facts on power series. In Theorem 1 we prove that if a sequence of numbers has OGF (ordinary generating function) that has zero convergence radius and satisfies a certain differential equation, then the sequence is far from being P-recursive. In Theorem 2 we apply this criterion to the sequences (con_n) and (cro_n). In Theorem 3 a more complicated differential equation is derived for the OGF of the sequence (sco_n). Finally, in Theorem 4 we show that modulo 2^l the sequences (con_n) and (cro_n) coincide with certain P-recursive, in fact algebraic, sequences.

Symbols $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers $\{\ldots,-1,0,1,\ldots\}$ and $\{1,2,\ldots\}$. $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real numbers and complex numbers. A sequence of complex numbers $(a_n)_{n\geq 0}$ is called *P-recursive* if there exist polynomials $P_0, P_1, \ldots, P_j \in \mathbb{C}[x]$, $P_0 \neq 0$, such that

$$P_0(n)a_n + P_1(n)a_{n-1} + \cdots + P_j(n)a_{n-j} = 0$$

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holds for each integer $n \geq j$. Many combinatorial counting sequences are P-recursive, for instance (mat_n), (ncr_n), Schröder numbers, and numbers of labelled k-regular graphs (Gessel [4]). But some are not, for instance Bell numbers and numbers of integer partitions.

We write $\mathbb{C}[[x_1, \ldots, x_k]]$ for the ring of power series with complex coefficients and variables $x_1, \ldots, x_k$. A power series $F \in \mathbb{C}[[x]]$ is *D-finite* if F solves the linear differential equation

$$R_0 F^{(m)} + R_1 F^{(m-1)} + \cdots + R_{m-1} F' + R_m F + R_{m+1} = 0$$

with polynomial coefficients $R_i \in \mathbb{C}[x]$, $R_0 \neq 0$. A sequence is P-recursive if and only if its OGF is D-finite. For the (easy) proof of this equivalence and further information and references on P-recursiveness and D-finiteness we refer to chapter 6 of Stanley [18]. At first these concepts were systematicly investigated in Stanley [17].

The ring $\mathbb{C}[[x_1, \ldots, x_k]]$ is contained in the ring of *Laurent series* $\mathbb{C}((x_1, \ldots, x_k))$ whose elements are formal sums

$$F = \sum_{i_1, \ldots, i_k \in \mathbb{Z}} a_{i_1, \ldots, i_k} x_1^{i_1} \ldots x_k^{i_k}$$

with $a_{i_1, \ldots, i_k} \in \mathbb{C}$ and such that, for some $p \in \mathbb{N}$ (depending on F), $(x_1 \ldots x_k)^p F \in \mathbb{C}[[x_1, \ldots, x_k]]$. For $k = 1$ the ring $\mathbb{C}((x_1, \ldots, x_k))$ is a field (but not for $k > 1$). We use notation

$$\begin{aligned} [x_1^{i_1} \ldots x_k^{i_k}]F &= a_{i_1, \ldots, i_k} \quad \text{and} \\ \mathrm{ord}_{x_j}(F) &= \min\{i_j \in \mathbb{Z} : \ a_{i_1, \ldots, i_k} \neq 0 \text{ for some } i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k \in \mathbb{Z}\}. \end{aligned}$$

By the definition, $\mathrm{ord}_{x_j}(F) > -\infty$. We set $\mathrm{ord}_{x_j}(0) = \infty$.

A power series $F \in \mathbb{C}[[x_1, \ldots, x_k]]$ is *analytic* if there is a real constant $\delta > 0$ such that the series $F(\alpha_1, \ldots, \alpha_k)$ absolutely converges for all $\alpha_i \in \mathbb{C}$, $|\alpha_i| < \delta$. Analytic power series form a ring denoted $\mathbb{C}\{x_1, \ldots, x_k\}$. This ring is closed also on division (if defined) and derivatives. Generalized coefficients of $F \in \mathbb{C}\{x_1, \ldots, x_k\}$ (obtained by distinguishing a subset of variables) are again analytic. For more details see Ruiz [15]. A Laurent series $F \in \mathbb{C}((x_1, \ldots, x_k))$ is analytic if, for some $p \in \mathbb{N}$, $(x_1 \ldots x_k)^p F \in \mathbb{C}\{x_1, \ldots, x_k\}$. Analytic Laurent series form again a ring closed on division, derivatives, and generalized coefficients. We say that $F \in \mathbb{C}[[x]]$ is *D_A-finite* if F solves the linear differential equation

$$R_0 F^{(m)} + R_1 F^{(m-1)} + \cdots + R_{m-1} F' + R_m F + R_{m+1} = 0$$

with analytic coefficients $R_i \in \mathbb{C}\{x\}$, $R_0 \neq 0$. We shall use the following fact.

Fact. If $G \in \mathbb{C}[[x, y]]$, $G(0, 0) = 0$, is nonzero and analytic, then every solution $F \in \mathbb{C}[[x]]$, $F(0) = 0$, of the equation

$$G(x, F) = 0$$

is analytic.

With the condition $\frac{\partial G}{\partial y}(0, 0) \neq 0$ this is the implicit function theorem for analytic functions (power series) and F is unique; see for example Hille [5] or Markushevich [10]. (Hille gives besides the analytic function approach also purely formal power series treatment of the theorem.) We could not find reference for this modest extension but we noticed that it follows from rather general results on analytic systems of equations in several variables treated in Ruiz [15]. By M. Artin's approximation theorem [15, p. 94], every solution of such a system can be approximated to any prescribed order by an analytic solution. Thus if an analytic system has a nonanalytic solution, it has infinitely many (analytic) solutions. But the equation $G(x, F) = 0$ is by Weierstrass preparation theorem [15, p. 14] equivalent to a polynomial equation $P(x, F) = 0$, where $P \in \mathbb{C}\{x\}[y]$, and therefore it has finitely many solutions. Hence all its solutions are analytic. See also [15, p. 106].

Theorem 1. *Let $F \in \mathbb{C}[[x]]$ satisfy differential equation*

$$F' = G(x, F) \tag{1}$$

where $G \in \mathbb{C}((x,y))$. Suppose that (i) F is not analytic, (ii) G is analytic, and (iii) $\mathrm{ord}_y(G) < 0$. *Then F is not D_A-finite, the less D-finite.*

Proof. Differentiating repeatedly (1) and substituting $G(x,F)$ for F', we express the derivatives of F as

$$F^{(k)} = G_k(x,F) \quad \text{where} \quad G_{k+1}(x,y) = \frac{\partial G_k}{\partial x} + G \cdot \frac{\partial G_k}{\partial y} \quad (\text{and } G_1 = G).$$

By condition (iii), $p = \mathrm{ord}_y(G) < 0$. Using condition (ii) and induction on k, we see that each $G_k \in \mathbb{C}((x,y))$ is analytic and

$$\mathrm{ord}_y(G_k) = k(p-1) + 1 < 0.$$

We suppose for contradiction that F is $\mathrm{D_A}$-finite and satisfies equation

$$R_0 F^{(m)} + R_1 F^{(m-1)} + \cdots + R_m F + R_{m+1} = 0$$

with $R_i \in \mathbb{C}\{x\}$, $R_0 \neq 0$. We may assume that $m > 0$ because $m = 0$ implies $F = -R_1/R_0 \in \mathbb{C}\{x\}$ and this contradicts condition (i). Replacing each $F^{(k)}$ by $G_k(x,F)$, we obtain equation

$$H(x,F) = 0 \quad \text{where} \quad H(x,y) = R_0(x)G_m(x,y) + \cdots + R_m(x)y + R_{m+1}(x).$$

Analyticity of G_k implies analyticity of H. It is clear that $H \neq 0$ because $\mathrm{ord}_y(H) = \mathrm{ord}_y(R_0 G_m) = m(p-1)+1 < 0$. Thus, by the above fact, $F \in \mathbb{C}\{x\}$, in contradiction with condition (i). □

This argument shows also that no power series $F + A$, $A \in \mathbb{C}\{x\}$, is $\mathrm{D_A}$-finite.

To the "non-Catalan" aspect of matching enumeration (many crossings, nonanalytic OGFs) are relevant articles (listed in chronological order) by Touchard [24, 25, 26], Kleitman [8], Hsieh [6], Riordan [14], Stein [19], Stein and Everett [20], Nijenhuis and Wilf [11], Read [13], Ismail, Stanton and Viennot [7], Penaud [12], Li and Sun [9], Cori and Marcus [2], Flajolet and Noy [3], Stoimenow [23], and Sawada [16]. We attempted to make this list of references complete or nearly complete.

Matchings are also called *complete pairings* or *linear(ized) chord diagrams*. In about 1993, their close relatives *(circular) chord diagrams* started to play, together with chord diagram algebras, an important role in studying knot invariants, especially Vassiliev knot invariants. By now more than 30 articles applying chord diagrams in knot theory have been published, of which we mention only Bar-Natan [1] and Stoimenow [21, 22]. Chord diagrams are orbits of the ($2n$-element) cyclic group acting on matchings by cyclically reordering vertices in $[2n]$. They appear in knot theory as follows. Suppose K is a singular knot, which is a smooth embedding of S^1 (1-circle) in $\mathbb{R}^3$ with n transversal selfintersections (double points). Fix an orientation of S^1 and associate with K a chord diagram X: mark on S^1 the $2n$ preimages of the selfintersections of K and connect by chord each two of them that correspond to one selfintersection. In fact, in these applications only those chord diagrams are important in which each chord crosses another chord (cf. cro_n). We refer to [1] for further information.

Recall that con_n and cro_n are numbers of connected and crossing matchings with n edges. We can rephrase their definition in terms of *crossing graphs*. For a matching X, this is a graph $G(X) = (X,E)$ with the vertex set X and the edge set $E = \{\{A,B\} : A,B \in X \text{ cross }\}$. Connected (crossing) matchings are matchings X for which $G(X)$ is connected (has no isolated vertex).

Theorem 2. *The power series*

$$E = \sum_{n\geq 1} \mathrm{con}_n x^n = x + x^2 + 4x^3 + 27x^4 + \cdots \quad \textit{and}$$
$$F = \sum_{n\geq 0} \mathrm{cro}_n x^n = 1 + x^2 + 4x^3 + 31x^4 + \cdots$$

satisfy differential equations

$$E' = \frac{E^2+E-x}{2xE} \quad \textit{and} \tag{2}$$

$$F' = \frac{-x^2F^3+F-1}{2x^3F^2+2x^2F}. \tag{3}$$

By Theorem 1, E and F are not D_A-finite. In particular, the sequences (con_n) and (cro_n) are not P-recursive.

Proof. By *spaces* in a matching X on $[2n]$ we mean the $2n-1$ *inner spaces* between the consecutive elements of $[2n]$ and the two spaces before and after $[2n]$. Alltogether X has $2n+1$ spaces. Let X be a connected matching on $[2n]$ with the first edge $\{1,a\} \in X$. Deleting it from X, the graph $G(X)$ falls apart into k components to which correspond nonempty matchings $X_1,\dots,X_k$. X_i has $n_i > 0$ edges and $X_1,\dots,X_k$ are "nested", which means that X_{i+1} lies in one of the $2n_i-1$ inner spaces of X_i. Also, a lies in one of the $2n_k-1$ inner spaces of X_k. There are no other restrictions on the X_is and a. Thus, in terms of E,

$$E = x\sum_{k\geq 0}(2xE'-E)^k.$$

Summing geometric series, we obtain equation (2).

Let X be a generic crossing matching on $[2n]$. We shall construct X by considering its edges which cross *only* the first edge $\{1,a\} \in X$. Suppose there are k such edges: $\{b_i,c_i\} \in X$, $b_1 < b_2 < \cdots < b_k < a < c_k < \cdots < c_2 < c_1$. The remaining edges of X lie either inside or outside each $\{b_i,c_i\}$ and the $n_i \geq 0$ edges sandwiched between $\{b_{i-1},c_{i-1}\}$ and $\{b_i,c_i\}$ form, for $i = 1,2,\dots,k+1$, a (possibly empty) matching X_i that is crossing (regardless of $\{1,a\}$).

We look first at case $k > 0$. X arises by choosing a crossing matching X_1 with $n_1 \geq 0$ edges, inserting $\{b_1,c_1\}$ in one of the $2n_1+1$ spaces of X_1, inserting a crossing matching X_2 inside $\{b_1,c_1\}$, inserting $\{b_2,c_2\}$ in one of the $2n_2+1$ spaces of X_2 (inside $\{b_1,c_1\}$), inserting a crossing matching X_3 inside $\{b_2,c_2\}$, and so on. In the end X_{k+1} is inserted inside $\{b_k,c_k\}$, 1 is put in the beginning, and a is inserted in one of the $2n_{k+1}+1$ spaces of X_{k+1} (and inside $\{b_k,c_k\}$, of course).

If $k = 0$, just $\{1,a\}$ is added. However, the difference is now that a cannot be inserted in each of the $2n_1+1$ spaces of X_1. Since $\{1,a\}$ must cross an edge, allowed are only spaces not separating X_1 into two crossing matchings. In terms of F,

$$F = 1+\sum_{k\geq 0}(x(2xF'+F))^{k+1} - xF^2.$$

Term xF^2 subtracts the number of spaces forbidden for a if $k = 0$. Summing geometric series, we obtain equation (3).

The conditions of Theorem 1 are easy to check. The right hand sides G of (2) and (3) are $2xy^{-1}(-x+y+y^2)$ and $2x^{-2}y^{-1}(-1+y-x^2y^3)(1+xy)^{-1}$, respectively. Thus both G belong to $\mathbb{C}(x,y)\cap\mathbb{C}((x,y))$ (are rational Laurent series) and for both $\mathrm{ord}_y(G) = -1$. Conditions (ii) and (iii) are satisfied. Clearly, $\mathrm{con}_n, \mathrm{cro}_n \geq (n-1)!$ (intersect one edge by the others). Thus condition (i) is satisfied as well and Theorem 1 applies. □

Writing (2) as $E = E(2xE'-E)+x$, we obtain the recurrence

$$\begin{aligned}\mathrm{con}_n &= \sum_{k=1}^{n-1}(2k-1)\cdot\mathrm{con}_k\cdot\mathrm{con}_{n-k}\\ &= (n-1)\sum_{k=1}^{n-1}\mathrm{con}_k\cdot\mathrm{con}_{n-k} \quad (n\geq 2 \text{ and } \mathrm{con}_1 = 1)\end{aligned}$$

found in [19] (it was derived by Stein and Riordan). A simple bijective proof of it was given in [11].

We give an alternative derivation for (2). Any matching X with $n > 0$ edges decomposes uniquely into a connected matching Y (the component of $G(X)$ containing the vertex corresponding to the first edge of X) with $m > 0$ edges and $2m$ matchings Z_i, $i = 1 \dots 2m$, which have l_i edges each and are inserted in the $2m+1$ spaces of Y except the first one. Each Z_i may be empty, no Z_i lies before Y, and $m + l_1 + \cdots + l_{2m} = n$. Thus, if E is the above OGF of (con_n) and $M = \sum_{n\geq 0} \mathrm{mat}_n x^n = 1 + x + 3x^2 + 15x^3 + \cdots$,

$$E(xM^2) = M - 1. \tag{4}$$

Formula $\mathrm{mat}_n = (2n-1)!!$ implies

$$M' = \frac{(1-x)M - 1}{2x^2}. \tag{5}$$

Here $\mathrm{ord}_y(G)$ is nonnegative, of course.

We write E_0 for $E(xM^2)$ and E_1 for $E'(xM^2)$. Thus (4) reads as $M = E_0 + 1$. Differentiating (4), solving the result for M', and replacing M by $E_0 + 1$, we obtain the second equation $M' = E_1(E_0+1)^2/(1 - 2xE_1(E_0+1))$. Substituting both expressions in (5), we obtain a polynomial equation $P(x, E_0, E_1) = 0$. Rewriting it as $Q(x(E_0+1)^2, E_0, E_1) = 0$ and substituting for x the power series inverse of $xM^2 = x(E_0+1)^2$, we get a polynomial equation $Q(x, E(x), E(x)') = 0$. It turns out to be (2).

This method may look complicated but its advantage is that it applies without much change to the natural generalization of the problem to r-matchings. These are partitions, for a fixed $r \geq 2$, of $\{1, 2, \dots, rn\}$ into r-element sets (in this article $r = 2$). Generalization of (non)crossing for $r > 2$ is easy (viz. noncrossing partitions). Formula for the number mat_n^r of all r-matchings and the analogue of (5) are straightforward. So is the analogue of (4). A differential equation for the OGF E of numbers con_n^r of connected r-matchings then can be derived by the method we have just described for $r = 2$. The equation involves derivatives of E to the order $r - 1$. The more combinatorial argument from the first part of the proof of Theorem 2 is still possible but for $r > 2$ becomes quite cumbersome.

From (4) also a relation between the numbers mat_n and con_n can be derived that, though standard, seems not to appear in the literature. Namely, writing M_0 for xM^2, (4) can be restated as $x(1 + E(M_0))^2 = M_0$ and Lagrange inversion formula gives $[x^n]M_0(x) = n^{-1}[x^{n-1}](1 + E(x))^{2n}$. In terms of the coefficients,

$$(n+1)\sum_{i=0}^{n}(2i-1)!! \cdot (2n-2i-1)!! = \sum \mathrm{con}_{a_1} \cdot \mathrm{con}_{a_2} \cdot \ldots \cdot \mathrm{con}_{a_{2n+2}}.$$

Here $(-1)!! = \mathrm{con}_0 = 1$ and the summation goes over all tuples such that $a_i \geq 0$ and $a_1 + \cdots + a_{2n+2} = n$.

Restating equation (3) as $F(1 + xF)(1 - xF - 2x^2F') = 1$ leads to the recurrence

$$\mathrm{cro}_n = \sum (2m+1) \cdot \mathrm{cro}_k \cdot \mathrm{cro}_l \cdot \mathrm{cro}_m \quad (n \geq 1 \ \text{ and } \ \mathrm{cro}_{-1} = \mathrm{cro}_0 = 1)$$

where we sum over $-1 \leq k, l, m \leq n-1$ & $k \neq -1$ & $k + l + m = n - 2$. This is a more explicit relation than the formulas in Stoimenow [23] on p. 217 (where cro_n is denoted as $\bar{\psi}_n$). The first few values of these numbers are:

$$\begin{aligned}(\mathrm{con}_n)_{n\geq 2} &= (1, 4, 27, 248, 2830, 38232, 593859, 10401712, \ldots)\\ (\mathrm{cro}_n)_{n\geq 2} &= (1, 4, 31, 288, 3272, 43580, 666143, 11491696, \ldots).\end{aligned}$$

The matchings X on $[2n]$ such that the graph $G(X)$ is connected and X does not change when the linear order of $[2n]$ is reversed were investigated first in [19]. Stein calls them *irreducible symmetric diagrams* and denotes their number (sco_n in our notation) as σ_{2n}.

Theorem 3. *The power series*

$$F = \sum_{n>0} \mathrm{sco}_n x^n = x + x^2 + 2x^3 + 7x^4 + \cdots$$

is related to the power series E of Theorem 2 by

$$F^2 - (1+x)F + E(x^2)\cdot(2xF' - F + 1) + x = 0 \tag{6}$$

and satisfies the second order differential equation

$$F''A_0 + (F')^3 A_1 + (F')^2 A_2 + F'A_3 + A_4 = 0 \tag{7}$$

where $A_i \in \mathbb{C}[x,F]$, namely $A_0 = -2x^2F^4 + (4x^2+4x^3)F^3 - (2x^2+8x^3+2x^4)F^2 + (4x^3+4x^4)F - 2x^4$, $A_1 = 8x^5$, $A_2 = 4x^2F^3 - (2x^2+6x^3)F^2 - (2x^2-4x^3+10x^4)F + 2x^3 + 10x^4$, $A_3 = -5xF^4 + (7x + 7x^2)F^3 + (x - 11x^2 + 4x^3)F^2 + (-3x + x^2 - 8x^3)F - x + 3x^2 + 4x^3$, and $A_4 = F^5 - (2+x)F^4 + (2x - x^2)F^3 + (2+3x^2)F^2 - (1 + 2x + 3x^2)F + x + x^2$.

Proof. The combinatorial argument leading to (6) is similar to those in the proof of Theorem 2 — remove the first and last edge and see what happens. It is somewhat more involved and in this extended abstract we omit it. The interested reader may want to try to find it as an exercise. Equation (7) is obtained by solving (6) for $E(x^2)$, differentiating the result, and substituting expressions for $E(x^2)$ and $E'(x^2)$ thus obtained in (2). □

Equating coefficients at x^n in (6) to zero leads to the formula

$$\begin{aligned} \mathrm{sco}_n \quad = \quad & \sum_{i=1}^{n-2} \mathrm{sco}_i \cdot \mathrm{sco}_{n-i} + \sum_{i=1}^{\lfloor n/2 \rfloor} (2n-4i-1)\cdot \mathrm{con}_i \cdot \mathrm{sco}_{n-2i} \\ & \text{where } n \geq 2,\ \mathrm{sco}_0 = -1, \text{and } \mathrm{sco}_1 = 1. \end{aligned}$$

This recurrence is simpler and more transparent than the one in the end of [19]. We have

$$(\mathrm{sco}_n)_{n\geq 2} = (1, 2, 7, 22, 96, 380, 1853, 8510, \ldots).$$

We say that a sequence (a_n) of integers is *P-recursive modulo* $m \in \mathbb{N}$ if there is a P-recursive sequence of integers (b_n) such that, for each n, $a_n \equiv b_n \bmod m$.

Theorem 4. *Sequences* (con_n) *and* (cro_n) *are P-recursive modulo* 2^k *for each* $k \in \mathbb{N}$.

Proof. We rewrite equation (2) for $E = \sum_{n>0} \mathrm{con}_n x^n$ as $E^2 + E - x = 2xEE'$. Differentiating it, replacing E' on the left by $(E^2 + E - x)/(2xE)$, and multiplying all by $2xE$, we obtain equation $2E^3 + 3E^2 + (1-4x)E - x = P_2$ where $P_2 \in 4\mathbb{Z}[x, E, E', E'']$. Continuing this way, we derive for each $k \in \mathbb{N}$ equation

$$Q_k(x, E) = P_k(x, E, E', \ldots, E^{(k)})$$

where Q_k and P_k are integral polynomials and P_k vanishes identically modulo 2^k. It is easy to see that the polynomial $[E^1]Q_k \in \mathbb{Z}[x]$ has always constant term 1. Thus the equation

$$Q_k(x, X) = 0$$

has a unique power series solution $A_k \in \mathbb{Z}[[x]]$, $Q_k(x, A_k) = 0$, with $A_k(0) = 0$. It follows by induction on n that $\mathrm{con}_n = [x^n]E \equiv [x^n]A_k \bmod 2^k$ for each n. A_k is algebraic over $\mathbb{Z}(x)$ and thus D-finite ([18], Theorem 6.4.6). Hence the coefficients of A_k form a P-recursive sequence and (con_n) is P-recursive modulo 2^k.

The proof for sequence (cro_n) is similar and we omit it. □

Lagrange inversion formula yields some explicit congruences. We give one example. Reducing (3) modulo 2 we obtain equation $x^2F^3 - F + 1 = 0$. With $G = xF$ it becomes $G = x(1-G^2)^{-1}$ and Lagrange inversion shows that cro_{2n+1} is always even and

$$\mathrm{cro}_{2n} \equiv \frac{1}{2n+1}\binom{3n}{n} \bmod 2.$$

The well known result that the highest power of 2 dividing $n!$ has exponent $\lfloor n/2 \rfloor + \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + \cdots$ gives a more explicit criterion: cro_{2n} is even if and only if there is an $r \in \mathbb{N}$ such that the residue of n on division by 2^r is greater than $2^{r+1}/3$. For example, cro_{304} is even (set $r = 5$) but cro_{296} is odd.

Conclusion. As we have indicated, the problem of enumeration of matchings with restricted crossings can be generalized to set partitions with r-element blocks (or to set partitions without any restriction on block sizes). However, then differential equations of more complicated types than (1) arise, such as (7). Probably, in these generalizations less amateur approach is necessary, with some techniques from differential algebra. Or maybe from geometry, as suggested by an anonymous referee. We conjecture that neither the sequence (sco_n) is P-recursive. Another research direction is to investigate the behaviour of con_n and cro_n with respect to other moduli. We conjecture that these numbers are P-recursive modulo m only if $m = 2^k$.

References

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