WITT'S FORMULA FOR RESTRICTED LIE ALGEBRAS

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ABSTRACT. Suppose that L is the free Lie algebra generated by k elements. This is a homogeneous algebra and the dimensions of its homogeneous components are given by Witt's formula $\dim L_n = \frac{1}{n}\sum_{a|n}\mu(a)k^{n/a}$, where $\mu(n)$ is the Möbius function. Recently the author found generating functions for the whole of the free Lie superalgebra. These series yield some new dimension formulas. Now we find generating functions for the free Lie p-algebras. As a corollary, we suggest dimension formulas similar to Witt's formula.

RÉSUMÉ.

1. INTRODUCTION: WITT'S FORMULA FOR FREE LIE SUPERALGEBRAS

Suppose that W is a superalgebra generated by sets $X_+ = \{x_1,\dots,x_m\}$, $X_- = \{x_{m+1},\dots,x_{m+k}\}$ and which is multihomogeneous with respect to these sets. For elements of W we introduce the *multidegree* $\alpha = (\alpha_1,\dots,\alpha_{m+k}) \in \mathbb{N}_0^{m+k}$, the *degree* $|\alpha| = \alpha_1+\cdots+\alpha_{m+k}$, and the numbers $|\alpha_+| = \alpha_1+\cdots+\alpha_m$, $|\alpha_-| = \alpha_{m+1}+\cdots+\alpha_{m+k}$, where $\mathbb{N}_0 = \{0,1,2,\dots\}$. By $a|\alpha$ we denote that a divides all components α_i. One has components for the gradation by the multidegree W_α, by the degree W_n, and by the *superdegree* W_{ij}, where the latter space consists of elements of degree i with respect to $x_1,\dots,x_m$, and degree j with respect to $x_{m+1},\dots,x_{m+k}$. Let $W_n = W_{n,+}\oplus W_{n,-}$ denote the decomposition into even and odd components, one also defines the *superdimension* as $\mathrm{sdim} W_n = \dim W_{n,+} - \dim W_{n,-}$. For a multihomogeneous subspace $V\subset W$ we define the following *Hilbert–Poincaré series*

$$V = \mathop{\oplus}_{\alpha\in\mathbb{N}_0^{m+k}} V_\alpha, \qquad \mathcal{H}(V,t_1,\dots,t_{m+k}) = \sum_{\alpha\in\mathbb{N}_0^{m+k}} \dim V_\alpha\, t_1^{\alpha_1}\cdots t_{m+k}^{\alpha_{m+k}};$$

$$V = \mathop{\oplus}_{i,j=0}^{\infty} V_{ij}, \qquad \mathcal{H}(V,t_+,t_-) = \sum_{i,j=0}^{\infty} \dim V_{ij}\, t_+^i t_-^j;$$

$$V = \mathop{\oplus}_{n=0}^{\infty} V_n, \qquad \mathcal{H}(V,t) = \sum_{n=0}^{\infty} \dim V_n\, t^n = \mathcal{H}(V,t_+,t_-)|_{t_+=t_-=t}.$$

The generating functions for the polynomial rings and the free associative algebras are well known. Let $W = K[X]$, $X = \{x_1,\dots,x_m\}$, be the polynomial ring and $W = K\langle X\rangle$, $X = \{x_1,\dots,x_m\}$ the free associative algebra. Then

$$\begin{aligned}\mathcal{H}(K[X],t_1,\dots,t_m) &= \frac{1}{(1-t_1)\cdots(1-t_m)};\\ \mathcal{H}(K\langle X\rangle,t_1,\dots,t_m) &= \frac{1}{1-(t_1+\cdots+t_m)}.\end{aligned} \tag{1}$$

These formulas have many applications. In particular they are applied to study invariants [10], [5]. In [12] the author found the generating functions for the free Lie (super)algebras. It seems that these formulas were not known before.

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Theorem 1.1 ([12]). *Let $L = L(X)$, $X = X_+ \cup X_-$ be the free Lie superalgebra where $X_+ = \{x_1, \dots, x_m\}$, $X_- = \{x_{m+1}, \dots, x_{m+k}\}$ are the even and odd generators, respectively. Then*

$$\mathcal{H}(L, t_1, \dots, t_m, t_{m+1}, \dots, t_{m+k}) =$$
$$= -\sum_{a=1}^{\infty} \frac{\mu(a)}{a} \ln\left(1 - t_1^a - \dots - t_m^a + (-t_{m+1})^a + \dots + (-t_{m+k})^a\right);$$
$$\mathcal{H}(L, t_+, t_-) = -\sum_{a=1}^{\infty} \frac{\mu(a)}{a} \ln\left(1 - m t_+^a + k(-t_-)^a\right);$$
$$\mathcal{H}(L, t) = -\sum_{a=1}^{\infty} \frac{\mu(a)}{a} \ln\Big(1 - (m - (-1)^a k) t^a\Big).$$

In the case of free Lie superalgebras we have formulas for dimensions of homogeneous and multihomogeneous components [2]. But these formulas include two cases, they correspond to the fact that the homogeneous component either consists only of regular words or it also contains squares of regular odd words. We suggest new formulas without two cases. See also a more general approach in generality of gradings by some semigroups [13].

Corollary 1.1 ([12]).

$$\dim L_n = \frac{1}{n} \sum_{a|n} \mu(a) (m - (-1)^a k)^{n/a};$$
$$\dim L_{n,+} = \frac{1}{n} \sum_{a|n} \mu(a) \frac{(m - (-1)^a k)^{n/a} + (m-k)^{n/a}}{2};$$
$$\dim L_{n,-} = \frac{1}{n} \sum_{a|n} \mu(a) \frac{(m - (-1)^a k)^{n/a} - (m-k)^{n/a}}{2};$$
$$\mathrm{sdim} L_n = \frac{1}{n} \sum_{a|n} \mu(a) \, (m-k)^{n/a}; \tag{2}$$
$$\dim L_\alpha = \frac{(-1)^{|\alpha_-|}}{|\alpha|} \sum_{a|\alpha} \mu(a) \frac{(|\alpha|/a)! \, (-1)^{|\alpha_-|/a}}{(\alpha_1/a)! \cdots (\alpha_{m+k}/a)!}. \tag{3}$$

The formula for the superdimensions (2) was obtained in [7]. Also, (3) was known for the case $X = X_-$ [9].

In [12] these formulas were proved directly. In [13] we consider another approach and a more general situation of the generating set $X = \cup_{\alpha\in\Gamma} X_\alpha$ being graded by a semigroup Γ, and instead of series we study characters for the whole of a superalgebra and for multihomogeneous components. As a particular case we consider the semigroup $\Gamma = \mathbb{N}^{m+k} \setminus \{0\}$ and monoid $\overline{\Gamma} = \mathbb{N}^{m+k}$. Denote $e_i = (\dots, 0, 1, 0, \dots)$, $i = 1, \dots, m+k$, with 1 on the ith place. We set $\varepsilon(e_1) = \dots = \varepsilon(e_m) = 1$, $\varepsilon(e_{m+1}) = \dots = \varepsilon(e_{m+k}) = -1$. We obtain the homomorphism $\varepsilon : \overline{\Gamma} \to \{\pm 1\}$ and the decomposition into the even and odd components $\Gamma = \Gamma_+ \cup \Gamma_-$, $\Gamma_+ = \{\alpha \in \Gamma \mid \varepsilon(\alpha) = 1\}$, $\Gamma_- = \{\alpha \in \Gamma \mid \varepsilon(\alpha) = -1\}$. We consider Lie superalgebras L that are decomposed into the even and odd components as follows $L = L_+ \oplus L_-$, $L_+ = \oplus_{\alpha\in\Gamma_+} L_\alpha$ $L_- = \oplus_{\alpha\in\Gamma_-} L_\alpha$. We denote $\mathbb{Q}[[\mathbf{t}]] = \mathbb{Q}[[t_1, \dots, t_{m+k}]]$ and let $\mathbb{Q}[[\mathbf{t}]]_0$ be the series without a constant term. An arbitrary series from $\mathbb{Q}[[\mathbf{t}]]$ we write as $\phi(\mathbf{t})$. Suppose that $V = \oplus_{\alpha\in\overline{\Gamma}} V_\alpha$ is a $\overline{\Gamma}$-graded vector space, then we also denote $\mathcal{H}(V, \mathbf{t}) = \mathcal{H}(V, t_1, \dots, t_{m+k})$.

We introduce two important operators as follows.

$$\mathcal{E}(\phi)(\mathbf{t}) = \exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\phi\Big|_{(t_i = \varepsilon(t_i)^{m+1}t_i^m)}\right), \qquad \phi(\mathbf{t}) \in \mathbb{Q}[[\mathbf{t}]]_0;$$

$$\mathcal{L}(\phi)(\mathbf{t}) = \sum_{a=1}^{\infty}\frac{\mu(a)}{a}\ln\left(\phi\Big|_{(t_i = \varepsilon(t_i)^{a+1}t_i^a)}\right), \qquad \phi(\mathbf{t}) \in 1+\mathbb{Q}[[\mathbf{t}]]_0.$$

The application of these operators is based on the following fact.

Theorem 1.2 ([13]). *Let $L = \oplus_{\alpha\in\Gamma}L_\alpha$ be a Γ-graded Lie superalgebra, $\Gamma = \mathbb{N}^{m+k}\setminus\{0\}$. Then the Hilbert-Poincaré series for L and its universal enveloping algebra $U(L)$ are connected as follows:*

1. $\mathcal{H}(U(L),\mathbf{t}) = \mathcal{E}(\mathcal{H}(L,\mathbf{t}))$;
2. $\mathcal{H}(L,\mathbf{t}) = \mathcal{L}(\mathcal{H}(U(L),\mathbf{t}))$.

Let $L = L(X)$ be the free Lie superalgebra generated by $X = X_+\cup X_-$, where $X_+ = \{x_1,\dots,x_m\}$ and $X_- = \{x_{m+1},\dots,x_{m+k}\}$ are the even and odd generators. It is well-known that $U(L)\cong K{<}X{>}$ [2]. Now using (1) and the second formula one can easily derive the formula for $\mathcal{H}(L,\mathbf{t})$ given by Theorem 1.1.

2. Witt's formula for free restricted Lie algebras

Let L be a Lie algebra over a field K of characteristic $p>0$. We use standard notation $\operatorname{ad} x : L\to L$, $\operatorname{ad} x(y) = [x,y]$, $x,y\in L$. Recall that L is called the *restricted Lie algebra* or *Lie p-algebra*, [6] if it additionally affords a unary operation $x\mapsto x^{[p]}$, $x\in L$, satisfying

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, when $\lambda\in K$, $x\in L$;
2. $\operatorname{ad}(x^{[p]}) = (\operatorname{ad} x)^p$;
3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$; $x,y\in L$, where $is_i(x,y)$ is the coefficient of t^{i-1} in the polynomial $\operatorname{ad}(tx+y)^{p-1}(x)\in L[t]$.

This notion is motivated by the following construction. Let A be an associative algebra over K, $\operatorname{char} K = p > 0$. The algebra A is turned into a Lie algebra by the new operation $[x,y] = xy - yx$, where $x,y\in A$. Then the mapping $x\mapsto x^p$, $x\in A$ satisfies these three properties.

Suppose that L is a restricted Lie algebra. Let J be the ideal in the universal enveloping algebra $U(L)$ generated by all elements $x^{[p]} - x^p$, $x\in L$. Then $u(L) = U(L)/J$ is called the *restricted enveloping algebra*. In this algebra the formal operation $x^{[p]}$ coincides with the pth power x^p for all $x\in L$. Let $\{a_i \mid i\in I\}$ be an ordered basis for L. By an analogue of Poincaré-Birkhoff-Witt's theorem [6], the restricted enveloping algebra has the following canonical basis:

$$u(L) = \langle a_{q_1}^{n_1}\cdots a_{q_s}^{n_s} \mid q_1 < \cdots < q_s, 0\le n_i < p\rangle_K.$$

Recall that a function of a natural argument $\alpha(n)$ is called *multiplicative* if $\alpha(nm) = \alpha(n)\alpha(m)$ for any coprime n,m. We introduce the following two functions

$$1_p(n) = \begin{cases} 1, & (p,n) = 1;\\ 1-p, & (p,n) = p.\end{cases}$$

$$\mu_p(n) = \begin{cases} \mu(n), & (p,n) = 1;\\ \mu(m)(p^s - p^{s-1}), & n = mp^s,\ (p,m) = 1,\ s\ge 1.\end{cases}$$

It is easy to check that these functions are multiplicative. One may view $\mu_p(n)$ as a deformation of the Möbius function.

By analogy with ordinary Lie algebras, one defines the free restricted Lie algebra L [1]. The goal of this paper is to find generating functions and dimension formulas for L. These formulas are completely analogous to those for the free Lie superalgebras.

Theorem 2.1. *Let $L = L_p(X)$ be the free Lie p-algebra generated by $X = \{x_1, \dots, x_m\}$. Then*

$$\begin{aligned}\mathcal{H}(L, t_1, \dots, t_m) &= -\sum_{a=1}^{\infty} \frac{\mu_p(a)}{a} \ln(1 - t_1^a - \dots - t_m^a);\\ \mathcal{H}(L, t) &= -\sum_{a=1}^{\infty} \frac{\mu_p(a)}{a} \ln(1 - mt^a).\end{aligned}$$

Corollary 2.1. *Let $L = L_p(X)$ be the free Lie p-algebra generated by $X = \{x_1, \dots, x_m\}$. Then*

$$\begin{aligned}\dim L_n &= \frac{1}{n} \sum_{a|n} \mu_p(a) m^{n/a};\\ \dim L_\alpha &= \frac{1}{|\alpha|} \sum_{a|\alpha_i} \mu_p(a) \frac{(|\alpha|/a)!}{(\alpha_1/a)! \cdots (\alpha_m/a)!}.\end{aligned}$$

It is possible to prove these formulas directly. Suppose that $L(X)$ is the free Lie algebra generated by X. One has the natural embedding $L(X) \subset U(L(X)) \cong K{<}X{>}$. Let $L_p(X) = \langle v^{[p^n]} \mid v \in L, n \geq 0 \rangle_K$, then $L_p(X)$ is the free restricted Lie algebra generated by X and $u(L_p(X)) \cong K{<}X{>}$ [2]. Also one has the following construction of a basis for $L_p(X)$. Suppose that $\{w_\alpha \mid \alpha \in \Lambda\}$ is a homogeneous basis for $L(X)$, then $\{w_\alpha^{[p^n]} \mid \alpha \in \Lambda, n \geq 0\}$ is a basis for the free p-algebra $L_p(X)$ [2]. This observation leads to the formula

$$\mathcal{H}(L_p(X), \mathbf{t}) = \sum_{n=0}^{\infty} \mathcal{H}\Big(L(X) \Big|_{(t_i = t_i^{p^n})} \Big).$$

Now it is possible to use Theorem 1.1 and Corollary 1.1. But let us consider another method. We suggest modification of operators $\mathcal{E}$, $\mathcal{L}$ as follows.

$$\begin{aligned}\mathcal{E}_p(\phi)(\mathbf{t}) &= \exp\left(\sum_{m=1}^{\infty} \frac{1_p(m)}{m} \phi \Big|_{(t_i = t_i^m)} \right), \qquad \phi(\mathbf{t}) \in \mathbb{Q}[[\mathbf{t}]]_0;\\ \mathcal{L}_p(\phi)(\mathbf{t}) &= \sum_{a=1}^{\infty} \frac{\mu_p(a)}{a} \ln\left(\phi \Big|_{(t_i = t_i^a)} \right), \qquad \phi(\mathbf{t}) \in 1 + \mathbb{Q}[[\mathbf{t}]]_0.\end{aligned}$$

Lemma 2.1. *The mappings $\mathcal{E}_p : \mathbb{Q}[[\mathbf{t}]]_0 \to 1 + \mathbb{Q}[[\mathbf{t}]]_0$ and $\mathcal{L}_p : 1 + \mathbb{Q}[[\mathbf{t}]]_0 \to \mathbb{Q}[[\mathbf{t}]]_0$ are well-defined and mutually inverse.*

Proof is based on the property $\sum_{ab=n} 1_p(b)\mu_p(a) = 0$ for all $n > 1$, that resembles the property of the Möbius function $\sum_{ab=n} \mu(a) = 0$, $n > 1$.

Now suppose that L is an $\mathbb{N}^m \setminus \{0\}$-graded restricted Lie algebra, here we additionally suppose that for all $x \in L_\alpha$, $\alpha \in \mathbb{N}^m \setminus \{0\}$ we have $x^{[p]} \in L_{p\alpha}$.

Theorem 2.2. *Let $L = \oplus_{\alpha \in \Gamma} L_\alpha$ be a Γ–graded restricted Lie algebra, $\Gamma = \mathbb{N}^m \setminus \{0\}$. Then the Hilbert-Poincaré series for L and its restricted enveloping algebra $u(L)$ are connected as follows:*

1. *$\mathcal{H}(u(L), \mathbf{t}) = \mathcal{E}_p(\mathcal{H}(L, \mathbf{t}))$;*
2. *$\mathcal{H}(L, \mathbf{t}) = \mathcal{L}_p(\mathcal{H}(u(L), \mathbf{t}))$.*

Let $L = L_p(X)$ be the free restricted Lie algebra generated by $X = \{x_1, \dots, x_m\}$. It is well-known that $u(L) \cong K{<}X{>}$ [2]. Using (1) and the second formula we immediately derive the formula for $\mathcal{H}(L, \mathbf{t})$.

$$\begin{aligned}\mathcal{H}(L, \mathbf{t}) &= \mathcal{L}_p(\mathcal{H}(u(L), \mathbf{t})) = \mathcal{L}_p(\mathcal{H}(K{<}X{>}, \mathbf{t})) = \mathcal{L}_p \frac{1}{1 - t_1 - \dots - t_m}\\ &= -\sum_{a=1}^{\infty} \frac{\mu_p(a)}{a} \ln(1 - t_1^a - \dots - t_m^a).\end{aligned}$$

Remark. One may consider the generality of *restricted Lie superalgebras* (see for definitions [2]). In this case analogous formulas are not so nice.

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