LITTLEWOOD'S MULTIPLE FORMULA FOR SPIN REPRESENTATIONS OF SYMMETRIC GROUPS

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Abstract. This note deals with some character values of the symmetric group as well as its double cover. Some observations of characters of these groups lead to what we call "Littlewood's Multiple Formula". The formula for the symmetric groups is found in Littlewood's book. We first give a simple proof of the formula. Our main objective is the spin character version of the formula which can be proved similarly to the original one.

0. Introduction

This note deals with some character values of the symmetric group S_n as well as its double cover $\tilde{S}_n$.

Let $\chi^\lambda(\rho)$ be the irreducible character of S_n, indexed by the partition λ and evaluated at the conjugacy class ρ. Comparing the character tables of S_2 and S_4,

	(1^2)	(2)
(2)	1	1
(1^2)	1	-1

	(2^2)	(4)
(4)	1	1
(2^2)	2	0

one observes that

$$\begin{aligned} \chi^{(4)}(2\rho) &= \chi^{(2)}(\rho), \\ \chi^{(2^2)}(2\rho) &= \chi^{(2)}(\rho) + \chi^{(1^2)}(\rho) \end{aligned}$$

for $\rho = (2)$, $2\rho = (4)$ and $\rho = (1^2)$, $2\rho = (2^2)$. A number of such observations lead to what we call "Littlewood's multiple formula" (Theorem 1.1). Although this formula appears in D. E. Littlewood's book [2], it does not seem to be a *folktale*. To be self-contained we include a proof of it, which is based on an "inflation" of the variables in a Schur function.

Our objective is the spin character version of Littlewood's multiple formula (Theorem 2.3). Let $\zeta^\lambda(\rho)$ be the irreducible negative character of $\tilde{S}_n$, indexed by the strict partition λ and evaluated at the conjugacy class ρ. One finds character tables $(\zeta^\lambda(\rho))$ in [1] for $n \le 14$.

	$(3,1)$	(1^4)
(4)	2	1
$(3,1)$	4	-1

	$(9,3)$	(3^4)
(12)	2	1
$(9,3)$	4	-1

This time phenomena observed are evident:

$$\zeta^{3\lambda}(3\rho) = \zeta^\lambda(\rho)$$

for $\lambda = (4), (3,1)$ and $\rho = (3,1), (1^4)$. The proof of Theorem 2.3 is achieved in a fashion similar to the case of ordinary characters. Instead of a Schur function, we deal with Schur's P-function which is defined as a ratio of Pfaffians.

1. Littlewood's multiple formula(original version)

We first recall the "multiple formula" for ordinary characters of the symmetric groups which is due to D. E. Littlewood [2, Ch. 8].

Throughout this section we fix a positive integer r. Let $\lambda = (\lambda_1, \dots, \lambda_N)$ be a partition. We always assume that N is at least the length $l(\lambda)$ of λ. An $(r+1)$-tuple of partitions $(\lambda^c, \lambda[0], \dots, \lambda[r-1])$ is attached to λ ; λ^c is the r-core of λ and the collection $\lambda^* = (\lambda[0], \dots, \lambda[r-1])$ is the r-quotient of λ (cf. [4, p12]). The effect of changing N is to permute the $\lambda[k]$ cyclically. Since we will only consider the Littlewood-Richardson coefficients $LR^{\mu}_{\lambda[0],\dots,\lambda[r-1]}$, the ambiguity of choosing N can be ignored. Put $r\lambda = (r\lambda_1, \dots, r\lambda_N)$. We can easily verify that $(r\lambda)^c = \emptyset$ and $(r\lambda)[0] = (\lambda_r, \lambda_{2r}, \lambda_{3r}, \dots)$, $(r\lambda)[k] = (\lambda_{r-k}, \lambda_{2r-k}\lambda_{3r-k}, \dots)(1 \le k \le r-1)$. As is well known, the ordinary irreducible characters of the symmetric group S_n are parametrized by $P(n)$, the set of all partitions of n. Let $\chi^{\lambda}(\rho)$ denote the irreducible character of S_n indexed by λ, evaluated at the conjugacy class determined by the partition ρ.

Theorem 1.1. *(Littlewood's multiple formula)*

$$\chi^{r\lambda}(r\rho) = \sum_{\nu \in P(n)} LR^{\nu}_{(r\lambda)[0],(r\lambda)[1],\dots,(r\lambda)[r-1]} \chi^{\nu}(\rho)$$

for any partitions λ and ρ of n.

D. E. Littlewood proved in [3, pp340-342] a more general formula. However, in order to contrast the formula with the spin character case, we here deal only with this form. In the rest of this section we give a simple proof of the theorem by using Schur functions.

We denote $\boldsymbol{x}_N = (x_1, x_2, \dots, x_N)$, $\boldsymbol{x}_N^r = (x_1^r, x_2^r, \dots, x_N^r)$ and $\boldsymbol{x}_{N,r} = (x_1, x_2, \dots, x_{rN})$ with $x_{kN+i} = (-1)^k x_i$ $(0 \le k \le r-1, 1 \le i \le N)$, where $(-1) = \exp(2\pi\sqrt{-1}/r)$. We will call $\boldsymbol{x}_{N,r}$ the "r-inflation" of $\boldsymbol{x}_N$.

It is sufficient to show the case of $r = 2$. For the general case, the proof is similar to the case of $r = 2$.

Let $\delta_N = (N-1, N-2, \dots, 1, 0)$. Following [4, p40] we define the Schur function of variables $\boldsymbol{x}_N$, corresponding to the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ by

$$s_{\lambda}(\boldsymbol{x}_N) = \frac{\det A_{\lambda+\delta_N}(\boldsymbol{x}_N)}{\det A_{\delta_N}(\boldsymbol{x}_N)},$$

where

$$A_{\alpha}(\boldsymbol{x}_N) = (x_i^{\alpha_j})_{1 \le i,j \le N}$$

for a sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ of non-negative integers.

Theorem 1.2.

$$s_{2\lambda}(\boldsymbol{x}_{N,2}) = s_{(2\lambda)[0]}(\boldsymbol{x}_N^2) s_{(2\lambda)[1]}(\boldsymbol{x}_N^2)$$

for any partition $\lambda = (\lambda_1, \dots, \lambda_{2N})$.

Proof. For a sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ of non-negative integers, let denote by

$$X_N^{\alpha_j+N-j} = \begin{pmatrix} x_1^{\alpha_j+N-j} \\ x_2^{\alpha_j+N-j} \\ \vdots \\ x_N^{\alpha_j+N-j} \end{pmatrix},$$

the j-th column of the matrix $A_{\alpha+\delta_N}(\boldsymbol{x}_N)$. We first compute the numerator of $s_{2\lambda}(\boldsymbol{x}_{N,2})$.

$$\begin{aligned} &A_{2\lambda+\delta_{2N}}(\boldsymbol{x}_{N,2}) = ((-1)^{(i-1)(2-j)} X_N^{2\lambda_j+2N-j})_{1\le i\le 2, 1\le j\le 2N} \\ = &\begin{pmatrix} X_N^{2\lambda_1+2N-1} & X_N^{2\lambda_2+2N-2} & \dots & X_N^{2\lambda_{2N}} \\ (-1)^{2-1}X_N^{2\lambda_1+2N-1} & (-1)^{2-2}X_N^{2\lambda_2+2N-2} & \dots & X_N^{2\lambda_{2N}} \end{pmatrix}. \end{aligned}$$

Let $\tau \in S_{2N}$ be defined by

$$\tau(2i+j) = N(j-1) + (i+1) \quad (0 \le i \le N-1, 1 \le j \le 2).$$

Permuting columns of the above matrix according to τ, and setting $p = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$, we obtain

$$\begin{aligned} &\det A_{2\lambda+\delta_{2N}}(\boldsymbol{x}_{N,2}) \\ &= \operatorname{sgn}(\tau) \begin{vmatrix} 1_N & 1_N \\ (-1)^{2-1}1_N & (-1)^{2-2}1_N \end{vmatrix} \\ &\times \begin{vmatrix} p^{2-1} & 0 \\ 0 & p^{2-2} \end{vmatrix} \\ &\times \begin{vmatrix} A_{(2\lambda)[1]+\delta_N}(\boldsymbol{x}_N^2) & 0 \\ 0 & A_{(2\lambda)[0]+\delta_N}(\boldsymbol{x}_N^2) \end{vmatrix}. \end{aligned}$$

Thus we have

$$\begin{aligned} &\det A_{2\lambda+\delta_{2N}}(\boldsymbol{x}_{N,2}) \\ &= \operatorname{sgn}(\tau)|p|2^N \det\left(A_{(2\lambda)[0]+\delta_N}(\boldsymbol{x}_N^2)\right) \det\left(A_{(2\lambda)[1]+\delta_N}(\boldsymbol{x}_N^2)\right). \end{aligned} \tag{1}$$

Putting $\lambda = (0, \dots, 0)$, we get

$$\det A_{\delta_{2N}}(\boldsymbol{x}_{N,2}) = \operatorname{sgn}(\tau)|p|2^N \left(\det A_{\delta_N}(\boldsymbol{x}_N^2)\right)^2. \tag{2}$$

From (1) and (2) we see that

$$\begin{aligned} s_{2\lambda}(\boldsymbol{x}_{N,2}) &= \frac{\det A_{2\lambda+\delta_{2N}}(\boldsymbol{x}_{N,2})}{\det A_{\delta_{2N}}(\boldsymbol{x}_{N,2})} \\ &= \frac{\det A_{(2\lambda)[0]+\delta_{2N}}(\boldsymbol{x}_N^2) \det A_{(2\lambda)[1]+\delta_{2N}}(\boldsymbol{x}_N^2)}{\det A_{\delta_N}(\boldsymbol{x}_N^2) \det A_{\delta_N}(\boldsymbol{x}_N^2)} \\ &= s_{(2\lambda)[0]}(\boldsymbol{x}_N^2) s_{(2\lambda)[1]}(\boldsymbol{x}_N^2). \end{aligned}$$

□

We remark here that the right-hand side of Theorem 1.2 can also be expressed as

$$s_{2\lambda[0]}(\boldsymbol{x}_N^2) s_{2\lambda[1]}(\boldsymbol{x}_N^2) = \sum_{\nu \in P(n)} LR^{\nu}_{(2\lambda)[0],(2\lambda)[1]} s_\nu(\boldsymbol{x}_N^2).$$

In order to translate the above identity of Schur functions into that of irreducible characters of S_n, we need the power sum symmetric functions

$$p_m(\boldsymbol{x}_N) = \sum_{i=1}^{N} x_i^m \quad (m = 1, 2, \dots).$$

For a partition $\rho = (\rho_1, \rho_2, \dots)$, define

$$p_\rho = p_{\rho_1} p_{\rho_2} \dots .$$

The well-known Frobenius formula reads

$$s_\lambda(\boldsymbol{x}_N) = \sum_{\rho \in P(n)} z_\rho^{-1} \chi^\lambda(\rho) p_\rho(\boldsymbol{x}_N),$$

where

$$z_\rho = \prod_{i=1}^{n} m_i!\ i^{m_i}$$

for the partition $\rho = (1^{m_1} 2^{m_2} \dots n^{m_n})$ of n. As for the r-inflation of $\boldsymbol{x}_N$, we have the following simple fact.

Lemma 1.3.

$$p_m(\boldsymbol{x}_{N,2}) = \begin{cases} 2p_m(\boldsymbol{x}_N) & m \equiv 0 (mod\ 2), \\ 0 & m \not\equiv 0 (mod\ 2). \end{cases}$$

Combining this lemma with the Frobenius formula we see that

$$\begin{aligned} s_{2\lambda}(\boldsymbol{x}_{N,2}) &= \sum_{\rho \in P(2n)} z_\rho^{-1} \chi^{2\lambda}(\rho) p_\rho(\boldsymbol{x}_{N,2}) \\ &= \sum_{\rho \in P(n)} z_{2\rho}^{-1} \chi^{2\lambda}(2\rho) 2^{l(\rho)} p_{2\rho}(\boldsymbol{x}_N) \\ &= \sum_{\rho \in P(n)} z_\rho^{-1} \chi^{2\lambda}(2\rho) p_\rho(\boldsymbol{x}_N^2). \end{aligned}$$

Therefore we have

$$\begin{aligned} & \sum_{\rho \in P(n)} z_\rho^{-1} \chi^{2\lambda}(2\rho) p_\rho(\boldsymbol{x}_N^2) \\ = & \sum_{\nu \in P(n)} LR^{\nu}_{(2\lambda)[0],(2\lambda)[1]} \left[\sum_{\rho \in P(n)} z_\rho^{-1} \chi^\nu(\rho) p_\rho(\boldsymbol{x}_N^2) \right] \\ = & \sum_{\rho \in P(n)} \left(\sum_{\nu \in P(n)} LR^{\nu}_{(2\lambda)[0],(2\lambda)[1]} \chi^\nu(\rho) \right) z_\rho^{-1} p_\rho(\boldsymbol{x}_N^2). \end{aligned}$$

Since p_ρ are linearly independent, we obtain

$$\chi^{2\lambda}(2\rho) = \sum_{\nu \in P(n)} LR^{\nu}_{(2\lambda)[0],(2\lambda)[1]} \chi^\nu(\rho)$$

as desired.

2. Spin Characters

We consider Littlewood's multiple formula for spin characters of symmetric groups. Theory of spin characters is that of Pfaffians by nature. Therefore Schur's P-functions play important roles. Here we adopt the definition of P-functions due to J. J. C. Nimmo [7, Appendix] (see also [4, p267]).

A partition $\lambda = (\lambda_1, \dots, \lambda_l)$ is said to be strict if $\lambda_1 > \dots > \lambda_l > 0$. The set of strict partitions of n is denoted by $SP(n)$.

Let

$$A(\boldsymbol{x}_N) = \left(\frac{x_i - x_j}{x_i + x_j} \right)_{1 \le i,j \le N},$$

and

$$D_\lambda(\boldsymbol{x}_N) = (x_i^{\lambda_j})_{1 \le i \le N,\ 1 \le j \le l}$$

for a strict partition $\lambda = (\lambda_1, \dots, \lambda_l)$. Put

$$A_\lambda(\boldsymbol{x}_N) = \begin{pmatrix} A(\boldsymbol{x}_N) & D_\lambda(\boldsymbol{x}_N) \\ -{}^tD_\lambda(\boldsymbol{x}_N) & 0 \end{pmatrix}$$

which is a skew-symmetric matrix of $N+l$ rows and columns. Define $\mathrm{Pf}_\lambda(\boldsymbol{x}_N)$ to be the Pfaffian of $A_\lambda(\boldsymbol{x}_N)$ if $N+l$ is even, and to be the Pfaffian of $A_\lambda(\boldsymbol{x}_{N+1})$ if $N+l$ is odd, agreeing $x_{N+1} = 0$. When $\lambda = \emptyset$, we have

$$\mathrm{Pf}_\emptyset(\boldsymbol{x}_N) = \begin{cases} \mathrm{Pf}A(\boldsymbol{x}_N), & \text{if } N \text{ is even,} \\ \mathrm{Pf}A(\boldsymbol{x}_{N+1}), & \text{if } N \text{ is odd.} \end{cases}$$

It is a good exercise to verify that

$$\mathrm{Pf}_\emptyset(\boldsymbol{x}_N) = \prod_{1 \le i < j \le N} \frac{x_i - x_j}{x_i + x_j}. \tag{3}$$

Now define Schur's P-function corresponding to the strict partition λ by

$$P_\lambda(\boldsymbol{x}_N) = \frac{\mathrm{Pf}_\lambda(\boldsymbol{x}_N)}{\mathrm{Pf}_\emptyset(\boldsymbol{x}_N)}.$$

Now we consider the 3−inflation of Schur's P−functions. The 3-inflation of $\boldsymbol{x}_N$ is defined by, as in Section 1, $\boldsymbol{x}_{N,3} = (x_1, x_2, \dots, x_{3N})$ with $x_{3N+i} = \omega^3 x_i$ $(0 \le k \le 2, 1 \le i \le N)$, where $\omega = \exp(2\pi\sqrt{-1}/3)$.

Lemma 2.1.

$$\mathrm{Pf}_\emptyset(\boldsymbol{x}_{N,3}) = c_3\{\mathrm{Pf}_\emptyset(\boldsymbol{x}_N^3)\}^3,$$

where c_3 is a non-zero complex number.

Proof. Put

$$b_k^+ = \prod_{1 \le i < j \le N} \frac{x_i - \omega^k x_j}{x_i + \omega^k x_j}, \quad b_k^- = \prod_{1 \le j < i \le N} \frac{x_i - \omega^k x_j}{x_i + \omega^k x_j},$$

and

$$d_k \;=\; (\frac{1-\omega^k}{1+\omega^k})^N$$

for $0 \le k \le r-1$. Then we check that

$$b_k^- = (-1)^{\frac{N(N-1)}{2}} b_{3-k}^+,$$

and

$$\begin{aligned} \prod_{k=0}^{2} b_k^+ &= \prod_{1 \le i < j \le N} \left(\frac{x_i - x_j}{x_i + x_j} \frac{x_i - \omega x_j}{x_i + \omega x_j} \frac{x_i - \omega^2 x_j}{x_i + \omega^2 x_j} \right) \\ &= \prod_{1 \le i < j \le N} \frac{x_i^3 - x_j^3}{x_i^3 + x_j^3} \\ &= \mathrm{Pf}_\emptyset(\boldsymbol{x}_N^3). \end{aligned}$$

By (3) we have

$$\mathrm{Pf}_\emptyset(\boldsymbol{x}_{N,3}) = c_3(b_0^+ b_1^+ b_2^+)^3 = c_3\{\mathrm{Pf}_\emptyset(\boldsymbol{x}_N^3)\}^3,$$

where

$$c_3 = (-1)^{3\frac{N(N-1)}{2}} d_1^2 d_2,$$

which is a non-zero constant. ∎

Before next theorem, we put

$$B_k = \left(\frac{x_i - \omega^k x_j}{x_i + \omega^k x_j}\right)_{1 \le i,j \le N} \quad (0 \le k \le 2).$$

Then we see that ${}^tB_0 = -B_0$ and ${}^tB_k = -B_{3-k}$ for $1 \le k \le 2$. We also see that

$$\mathrm{Pf}_{\emptyset}(\boldsymbol{x}_{N,2}) = \begin{vmatrix} B_0 & B_1 & B_2 \\ B_2 & B_0 & B_1 \\ B_1 & B_2 & B_0 \end{vmatrix}^{1/2},$$

if N is even, and

$$\mathrm{Pf}_{\emptyset}(\boldsymbol{x}_{N,3}) = \begin{vmatrix} B_0 & B_1 & B_2 & 1 \\ B_0 & B_0 & B_1 & 1 \\ B_1 & B_2 & B_0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix}^{1/2},$$

if N is odd.

Theorem 2.2.

$$P_{3\lambda}(\boldsymbol{x}_{N,3}) = P_\lambda(\boldsymbol{x}_N^3)$$

for any strict partition $\lambda = (\lambda_1, \dots, \lambda_l)$.

Proof. We give a proof for the case that $N + l$ is even, since the other case can be proved with only a slight modification. Set

$$D = D_\lambda(\boldsymbol{x}_N^3).$$

Since $3N + l$ is even, we have

$$\mathrm{Pf}_{r\lambda}(\boldsymbol{x}_{N,3}) = \begin{vmatrix} B_0 & B_1 & B_2 & D \\ B_2 & B_0 & B_1 & D \\ B_1 & B_2 & B_0 & D \\ -{}^tD & -{}^tD & -{}^tD & 0 \end{vmatrix}^{1/2}.$$

Apply the following elementary row-block and column-block oprations succesively to the above matrix:

1. Subtract the third row-block from the first and second row-blocks.
2. Subtract the third column-block from the first and second column-blocks .
3. Add the third row-block to the first and second low-blocks multiplied by 1/3 .
4. Add the third column-block to the first and second column-blocks multiplied by 1/3 .

This series of oprations preserves the skew-symmetricity of the matrix and does not change its Pfaffian. Therefore we have, setting $B = 1/3 \sum_{k=0}^{2} B_k$,

$$\begin{aligned} \mathrm{Pf}_{3\lambda}(\boldsymbol{x}_{N,3}) &= \begin{vmatrix} A' & 0 & 0 \\ 0 & B & D \\ 0 & -{}^tD & 0 \end{vmatrix}^{1/2} \\ &= \mathrm{Pf}(A') \begin{vmatrix} B & D \\ -{}^tD & 0 \end{vmatrix}^{1/2} \end{aligned}$$

for some skew-symmetric matrix A' of $3(N-1)$ rows and columns. By a simple calculation we can show that the (i,j)-entry of B is equal to

$$\frac{1}{3}\sum_{k=0}^{2} \frac{x_i - \omega^k x_j}{x_i + \omega^k x_j} = \frac{x_i^3 - x_j^3}{x_i^3 + x_j^3},$$

and thus we see that $\mathrm{Pf}(B) = \mathrm{Pf}_{\emptyset}(\boldsymbol{x}_N^3)$. Hence we have

$$\left| \begin{array}{cc} B & D \\ -{}^tD & 0 \end{array} \right|^{1/2} = \mathrm{Pf}_\lambda(\boldsymbol{x}_N^3).$$

By the same elementary operations as above we see that

$$\begin{aligned} \mathrm{Pf}_\emptyset(\boldsymbol{x}_{N,3}) &= \left| \begin{array}{cc} A' & 0 \\ 0 & B \end{array} \right|^{1/2} \\ &= \mathrm{Pf}(A')\mathrm{Pf}_\emptyset(\boldsymbol{x}_N^3). \end{aligned}$$

Since the left-hand side is equal to $c_r\{\mathrm{Pf}_\emptyset(\boldsymbol{x}_N^3)\}^3$, we have

$$\mathrm{Pf}(A') = c_3\{\mathrm{Pf}_\emptyset(\boldsymbol{x}_N^3)\}^2.$$

Consequently we see

$$\begin{aligned} P_{3\lambda}(\boldsymbol{x}_{N,3}) &= \frac{\mathrm{Pf}_{3\lambda}(\boldsymbol{x}_{N,3})}{\mathrm{Pf}_\emptyset(\boldsymbol{x}_{N,3})} \\ &= \frac{c_3\{\mathrm{Pf}_\emptyset(\boldsymbol{x}_N^3)\}^2\mathrm{Pf}_\lambda(\boldsymbol{x}_N^3)}{c_3\{\mathrm{Pf}_\emptyset(\boldsymbol{x}_N^3)\}^3} \\ &= P_\lambda(\boldsymbol{x}_N^3). \end{aligned}$$

□

Let $\tilde{S}_n$ $(n \geq 4)$ be the double cover of the symmetric group S_n, which is generated by $t_1, \dots, t_{n-1}$ and z subject to the relations:

$$\begin{cases} z^2 = 1, \\ zt_i = t_iz, \ t_i^2 = z \quad (1 \leq i \leq n-1), \\ (t_it_{i+1})^3 = z \quad (1 \leq i \leq n-2), \\ t_it_j = zt_jt_i \quad (|i-j| \geq 2). \end{cases}$$

The group $\tilde{S}_n$ can be regarded as a central extention

$$1 \longrightarrow Q \longrightarrow \tilde{S}_n \xrightarrow{\ \theta\ } S_n \longrightarrow 1,$$

where $Q = \{1, z\} \cong S_2$. The central element z acts on an irreducible representation by ± 1. An irreducible representation of $\tilde{S}_n$ is said to be negative if $z = -1$. It is known by I. Schur (cf. [1, Ch. 8]) that, roughly speaking, the irreducible negative representations of $\tilde{S}_n$ are parametrized by $SP(n)$. We will denote by ζ^λ the character of the irreducible negative representation corresponding to $\lambda \in SP(n)$. These spin characters are related with the P-functions by the following Schur formula ([6, p64]):

$$P_\lambda(\boldsymbol{x}_N) = \sum_{\rho \in P_{odd}(n)} 2^{(l(\rho)-l(\lambda)+\epsilon(\lambda))/2} z_\rho^{-1} \zeta^\lambda(\rho) p_\rho(\boldsymbol{x}_N),$$

where $P_{odd}(n)$ denotes the set of partitions of n consisting of odd parts, $\zeta^\lambda(\rho)$ is the value of ζ^λ on the conjugacy class determined by $\rho \in P_{odd}(n)$, and

$$\epsilon(\lambda) = \begin{cases} 0, & \text{if } n - l(\lambda) \text{ is even}, \\ 1, & \text{if } n - l(\lambda) \text{ is odd}. \end{cases}$$

Using the Schur formula we can translate Theorem 2.2 into the following multiple formula for spin characters.

Theorem 2.3.
$$\zeta^{3\lambda}(3\rho) = \zeta^{\lambda}(\rho)$$
for any $\lambda \in SP(n)$ *and* $\rho \in P_{odd}(n)$.

Proof. We have
$$\begin{aligned}
P_{3\lambda}(\boldsymbol{x}_{N,3}) &= \sum_{\rho \in P_{odd}(3n)} 2^{(l(\rho)-l(3\lambda)+\epsilon(3\lambda))/2} z_\rho^{-1} \zeta^{3\lambda}(\rho) p_\rho(\boldsymbol{x}_{N,3}) \\
&= \sum_{\rho \in P_{odd}(n)} 2^{(l(3\rho)-l(3\lambda)+\epsilon(3\lambda))/2} z_{3\rho}^{-1} \zeta^{3\lambda}(3\rho) 3^{l(\rho)} p_\rho(\boldsymbol{x}_N^3) \\
&= \sum_{\rho \in P_{odd}(n)} 2^{(l(3\rho)-l(3\lambda)+\epsilon(3\lambda))/2} z_\rho^{-1} \zeta^{3\lambda}(3\rho) p_\rho(\boldsymbol{x}_N^3).
\end{aligned}$$
The left-hand side is equal to
$$P_\lambda(\boldsymbol{x}_N^3) = \sum_{\rho \in P_{odd}(n)} 2^{(l(\rho)-l(\lambda)+\epsilon(\lambda))/2} z_\rho^{-1} \zeta^{\lambda}(\rho) p_\rho(\boldsymbol{x}_N^3).$$
Since $l(3\rho) = l(\rho)$, $l(3\lambda) = l(\lambda)$ and $\epsilon(3\lambda) = \epsilon(\lambda)$, we can conclude, comparing the coefficients of $p_\rho(\boldsymbol{x}_N^3)$, that $\zeta^{3\lambda}(3\rho) = \zeta^{\lambda}(\rho)$. □

Remark 2.4. Theorem 2.3 is generalized following statement.
$$\zeta^{r\lambda}(r\rho) = \zeta^{\lambda}(\rho)$$
for any odd positive integer r, $\lambda \in SP(n)$ and $\rho \in P_{odd}(n)$.
c. f. [5,Theorem2.3].

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